Designing Automated Market Makers for Combinatorial Securities: A Geometric Viewpoint

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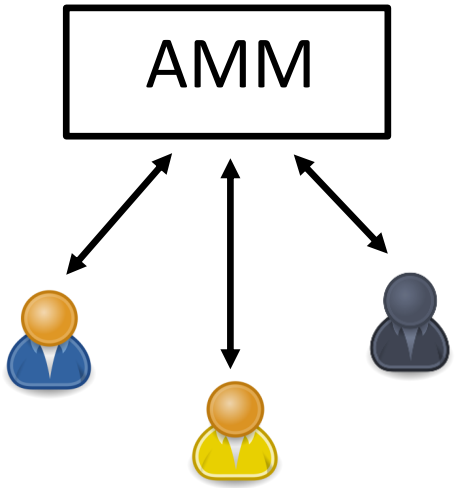
Outline

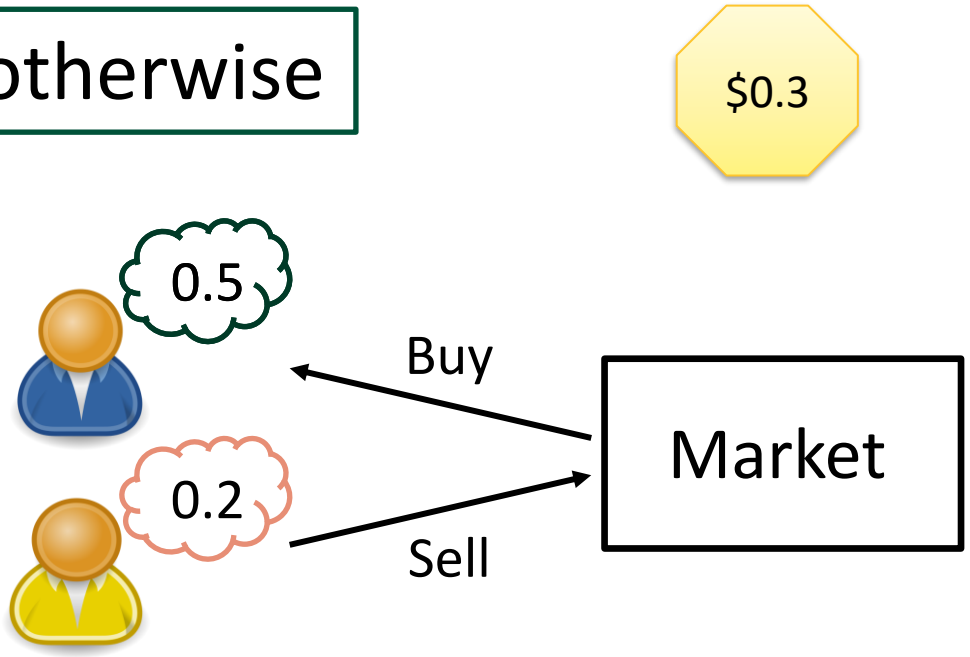
- Preliminary of prediction market
  - Automatic market maker and LSMR
  - Large outcome space: combinatorial securities
- AMM as a range query problem
- AMM for de-fi



Why Can Markets Aggregate Information?

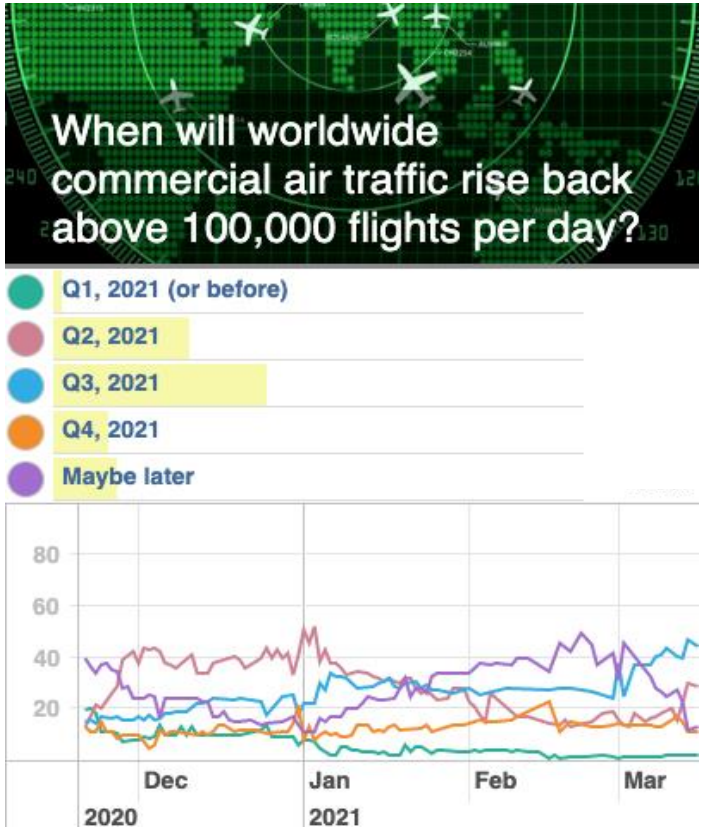
- Offer securities whose payoff is tied to outcomes of an event.

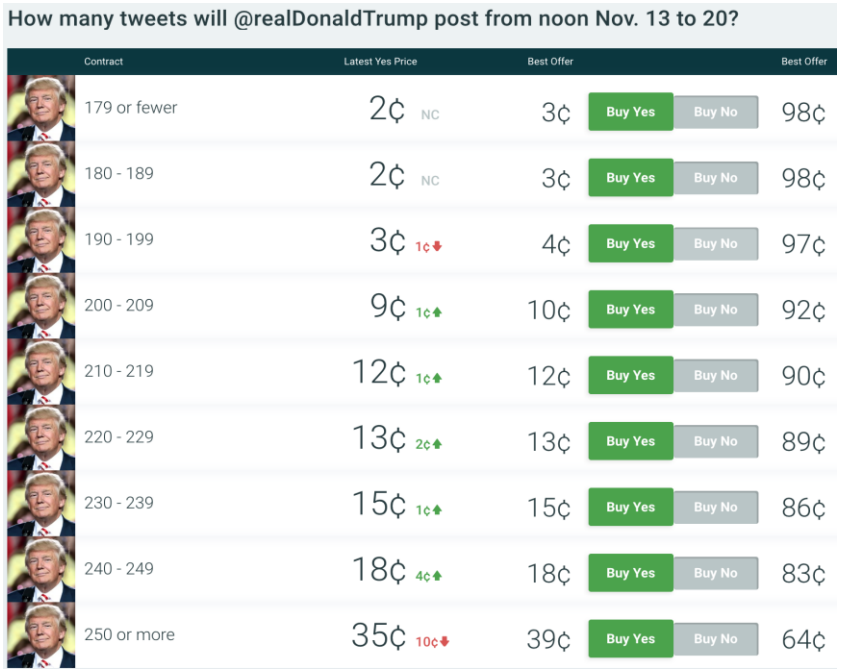
$1 if rain on Jan 15, $0 otherwise



  - A market maker adjust the price to "equilibrium"
  - Price ≈ Pr[event |all information]

Examples of Prediction Markets

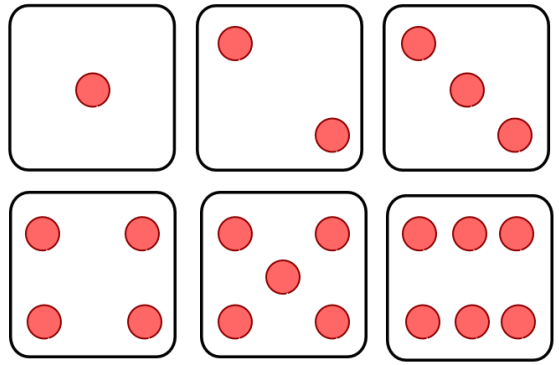




AMM for prediction market

- Shares $w = (w_1, \dots w_n)$
- Each security $i = 1, \dots, n$ has
  - An outcome and pay 1$ if happens
  - Price $p_i(w)$
- Experts: buy s shares of security i and pay
- Market maker (AMM): change prices based on shares

Security	1	2	3	4	5	6
Share w_i	1	1	1	1	2	2
Price p_i	1/8	1/8	1/8	1/8	1/4	1/4



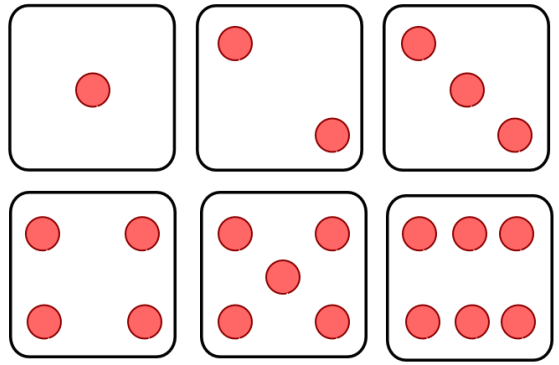
$n = 6$

Logarithmic Market Scoring Rule

Let $C(w) = \log_2 \sum_{i=1}^{n} 2^{w_i}$ and $w = (w_1, \dots w_n)$

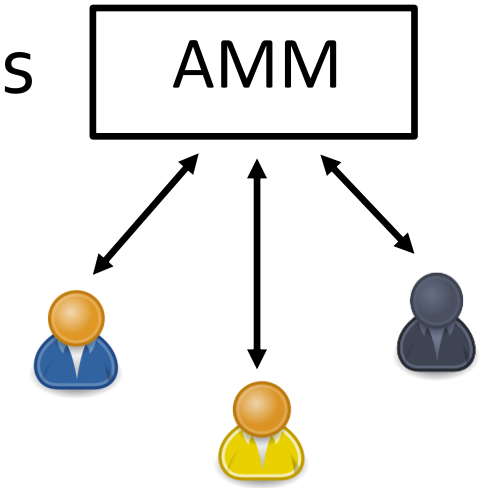
- Each security i has
  - An outcome and pay 1$ if happens
  - Price $p_i(w) = \frac{2^{w_i}}{\sum_j 2^{w_j}} = \frac{\partial C(w)}{\partial w_i}$
- Experts: buy s shares of security i and pay $C(w + s\delta_i) - C(w)$

Security	1	2	3	4	5	6
Share w_i	1	1	1	1	2	2
Weight 2^{w_i}	2	2	2	2	4	4
Price p_i	1/8	1/8	1/8	1/8	1/4	1/4



LMSR as an online algorithm problem

- Given a set of outcomes and initial shares $w^{(0)}$, in each round t
  1. Query price(i): return $p_i\left(w^{(t)}\right)$
  2. Query cost(i, s): return $C\left(w^{(t)} + s\delta_i\right) - C\left(w^{(t)}\right)$
  3. Query buy(i, s): update $w^{(t+1)} = w^{(t)} + s\delta_i$
- Design a data structure to support above queries



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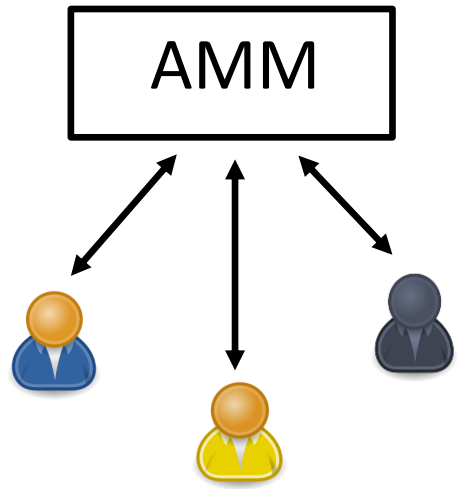
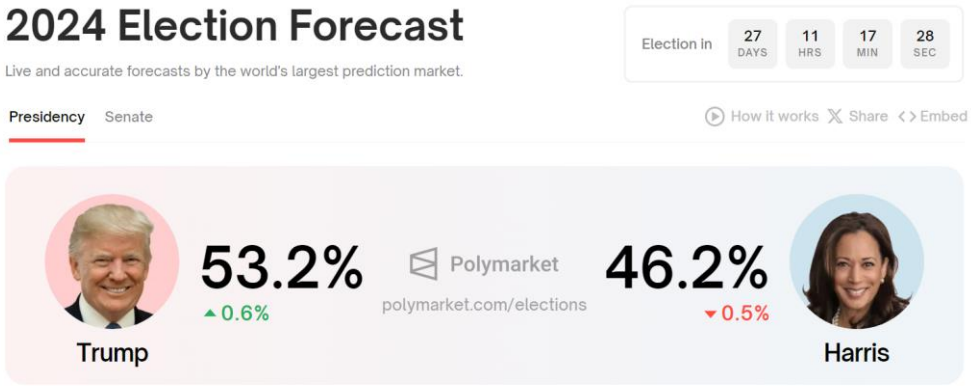


Prediction market

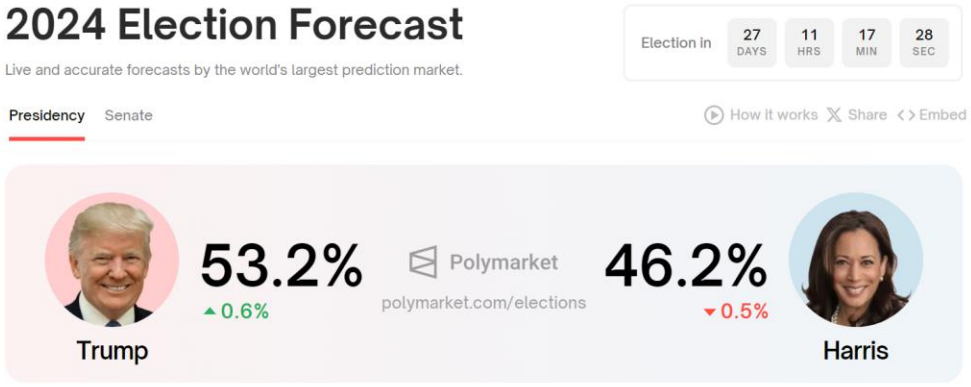
- Given a random variable on $\mathcal{X}$, trade securities with an expert
  - Each security has
    - An outcome and pay 1$ if happens
    - Price
  - Experts: buy securities
  - AMM: change prices based on shares

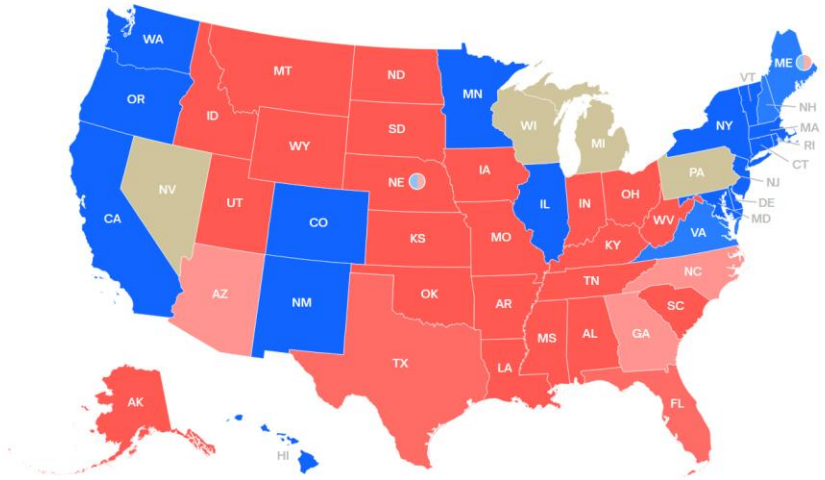


$\mathcal{X}$ ={Harris wins, Trump wins}

Large outcome space

- Given a random variable on $\mathcal{X}$, trade securities with an expert
  - Each security has
    - An outcome and pay 1$ if happens
    - Price
  - Experts: buy securities
  - AMM: change prices based on shares



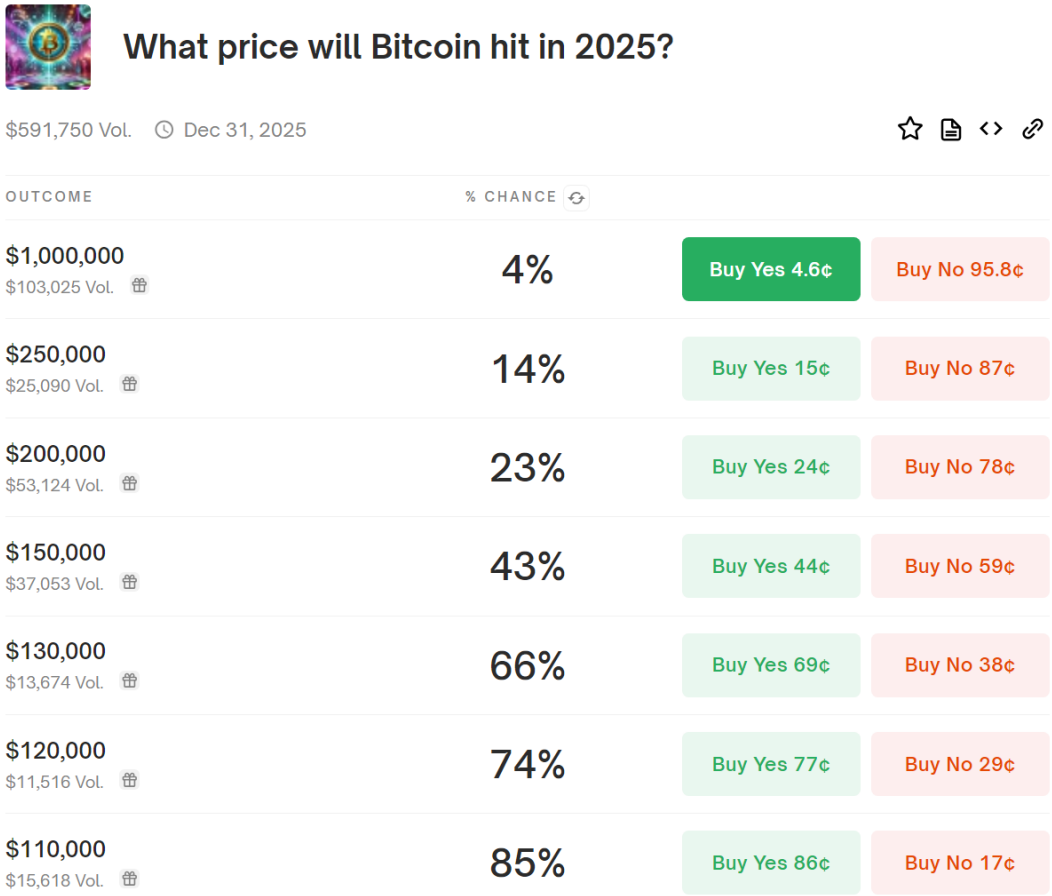


$\mathcal{X}$ ={Each state's winner}

Large outcome space

- Given a random variable on $\mathcal{X}$, trade securities with an expert
  - Each security has
    - An outcome and pay 1$ if happens
    - Price
  - Experts: buy securities
  - AMM: change prices based on shares

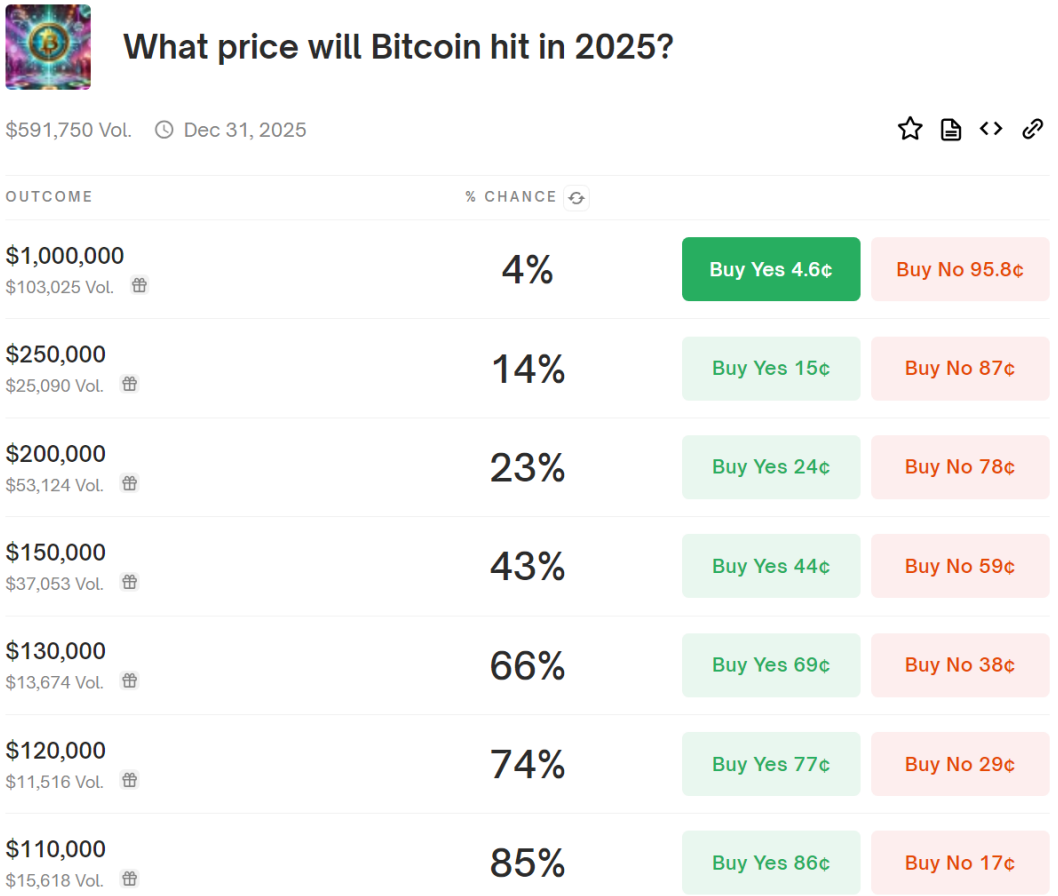
$$\mathcal{X} = \mathbb{R}$$



Combinatorial prediction market

- Given a random variable on $\mathcal{X}$, trade securities with an expert
  - Each security in $\mathcal{F}$ has
    - A **subset of outcomes** and pay 1$ if happens
    - Price
  - Experts: buy securities
  - AMM: change prices based on shares

$$\mathcal{X} = \mathbb{R}$$

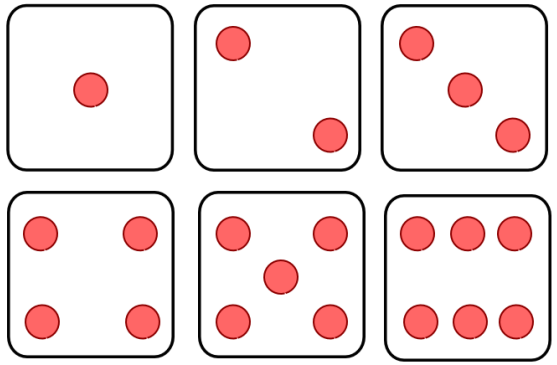


Examples of combinatorial securities $(\mathcal{X}, \mathcal{F})$

- Continuous
  - Bitcoin price
    - Intervals, e.g., $[110k, +\infty], [120k, +\infty]$
  - Stock market
    - 2D intervals: the opening value of AMZN and GOOGL at 4pm tomorrow
    - Hyperplane: Index funds (S&P500)
- Discrete
  - Presidential election
    - Swing states, Popular votes
  - Sport betting
    - Playoff, championship

Traditional Market Implementation

- Independent markets
  - Logic inconsistent
  - Arbitrage
- Integrated combinatorial prediction market for $(\mathcal{X}, \mathcal{F})$
  - Computational complexity [CFNP07]
  - Algorithms for Intervals [DWPR21]



Our Contributions

AMM for combinatorial securities = range query problem

- Partition tree algorithms for LMSR
- Generalize to quadratic scoring rule and 3/2-power scoring rule
- Incorporate multi-resolution market designs into a partition-tree scheme.
- Combinatorial Swap in DeFi

Set systems $(\mathcal{X}, \mathcal{F})$	Running time
Intervals (Example 2.3)	$\Theta(\log n)$
d-orthogonal sets (Example 2.4)	$O(n^{1-1/d})$
Hyperplane in $\mathbb{R}^d$ (Example 2.5)	$O(n^{1-1/d})$
Finite VC (Example 2.6)	$O(n^{1-\epsilon})$ with $\epsilon > 0$
Infinite VC (Examples 2.7 and 2.8)	no $o(n)$

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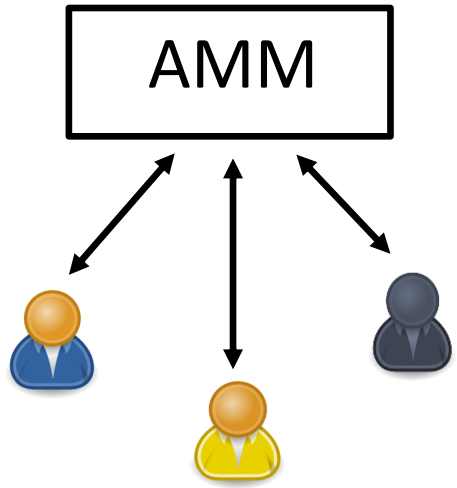
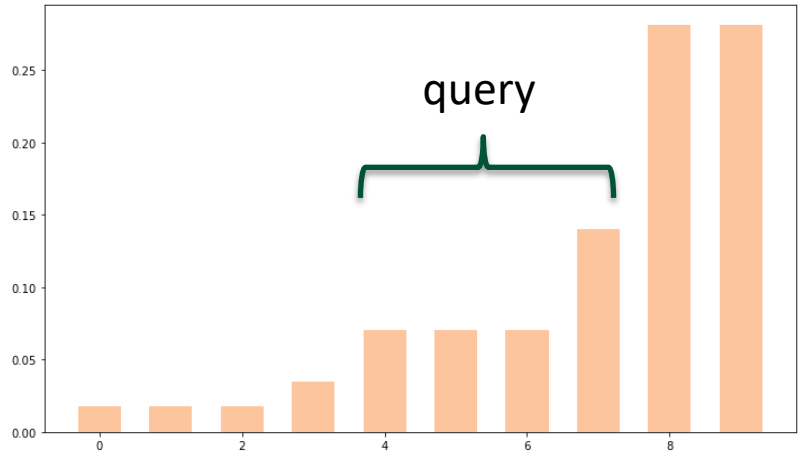
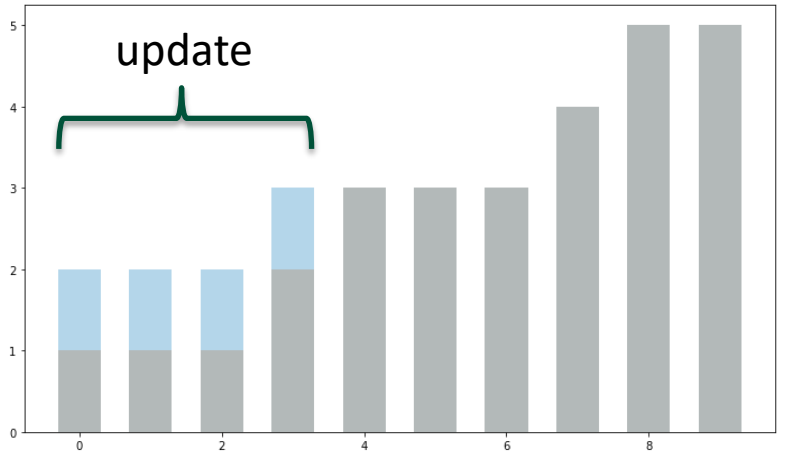


Range query range update (RQRU)

Given $(\mathcal{X}, \mathcal{F})$ and initial weights $W^{(0)}: \mathcal{X} \to \mathbb{R}_+$, RQRU requests a sequence of operations: for any $E \in \mathcal{F}$ and $S \in \mathbb{R}_+$:

- query$(E; W)$: return the total weight of range E, $\sum_{x \in E} W(x)$.
- update$(E, S; W)$: update $W(x) \leftarrow \begin{cases} S \cdot W(x) \text{ if } x \in E \\ W(x) \text{ otherwise} \end{cases}$





Idea

Given $\mathcal{F}$, $C(w) = \log_2 \sum_i 2^{w_i}$ and w

- Each security has
  - An event E in $\mathcal{F}$ and pay 1$ if happens
  - Price $p_E(w) = \sum_{i \in E} \frac{2^{w_i}}{\sum_j 2^{w_j}}$
- Experts: buy s shares of security E and pay $C(w + s1_E) - C(w)$

Idea: price=query

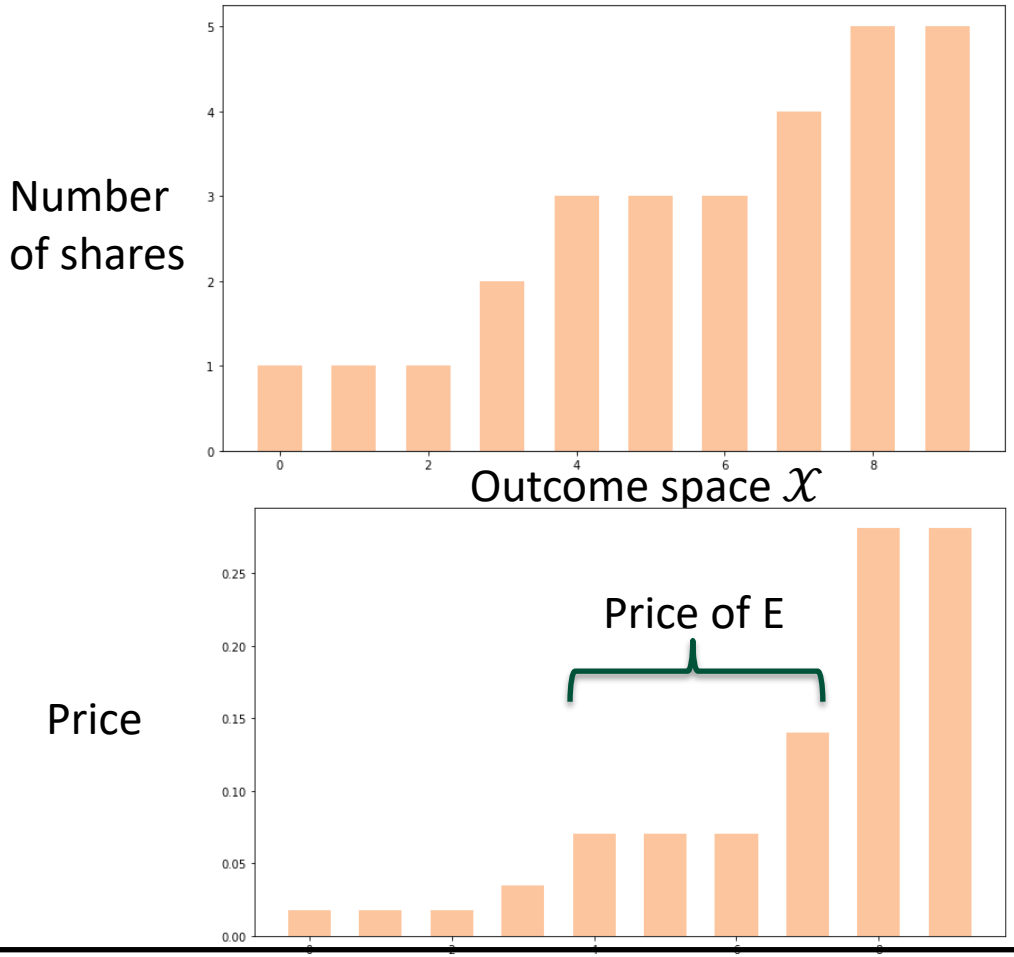
Given $\mathcal{F}$, $C(w) = \log_2 \sum_i 2^{w_i}$ and w

- Each security has
  - An event E in $\mathcal{F}$ and pay 1$ if happens

query

  - Price $p_E(w) = \sum_{i \in E} \frac{2^{w_i}}{\sum_j 2^{w_j}}$

- Experts: buy s shares of security E and pay $C(w + s1_E) - C(w)$

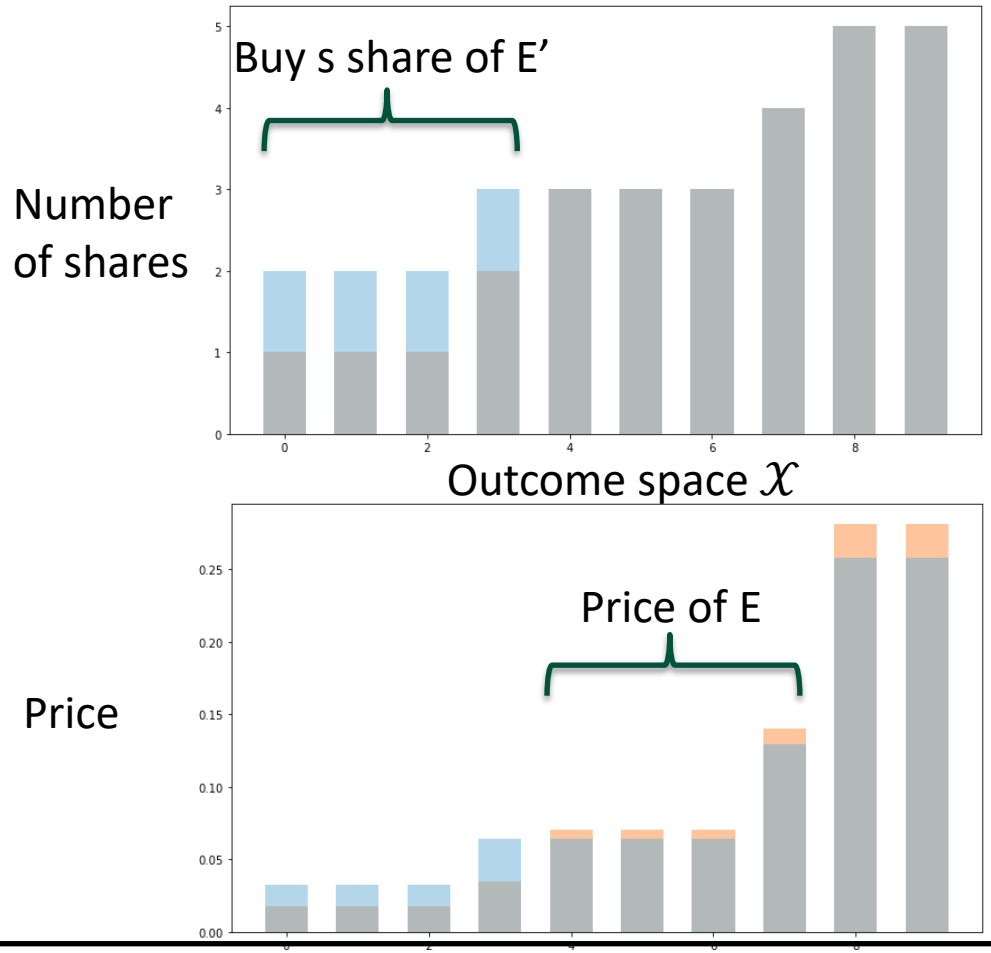


Idea: buy=update

Given $\mathcal{F}$, $C(w) = \log_2 \sum_i 2^{w_i}$ and w

- Each security has
  - An event E in $\mathcal{F}$ and pay 1$ if happens
  - Price $p_E(w) = \sum_{i \in E} \frac{2^{w_i}}{\sum_j 2^{w_j}}$
- Experts: buy s shares of security E and pay $C(w + s1_E) - C(w)$

update



AMM = range query problem

AMM

- Given $\mathcal{X}$ and securities $\mathcal{F}$, in each round
  - Experts: buy/sale securities $E \in \mathcal{F}$
  - Market maker: update prices

Range query problem

- Given $\mathcal{X}$ and ranges $\mathcal{F}$, each rounds
  - Update state of the space
  - Query sum of points in a range $E \in \mathcal{F}$

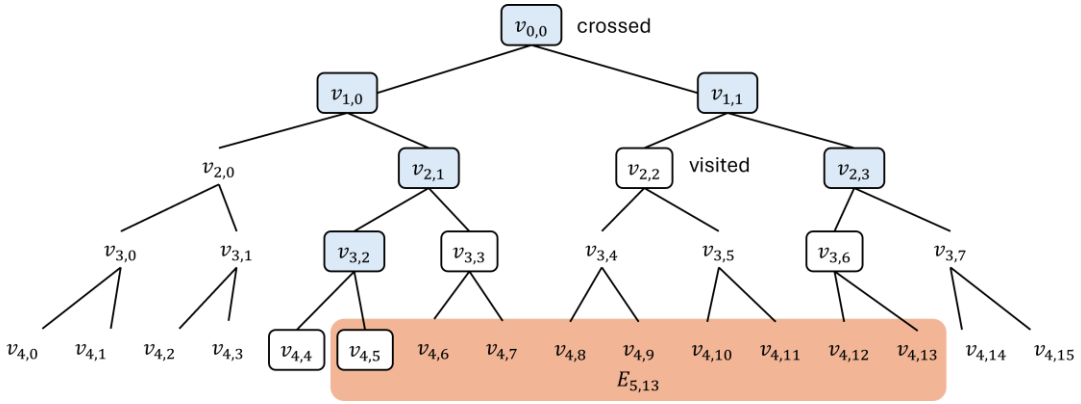
LMSR		RQRU
Price and cost	$w_i = 2^{w_i}$ $\longleftrightarrow$	Query (+)
Buy		Update (*)

Beyond LMSR

- AMM=RQRU

Scoring rule	Query	Update
Log scoring rule	Addition	Multiplication
Quadratic scoring rule	Addition	Addition
3/2-power scoring rule	Addition	Group action

– multi-resolution market: local arbitrage removal

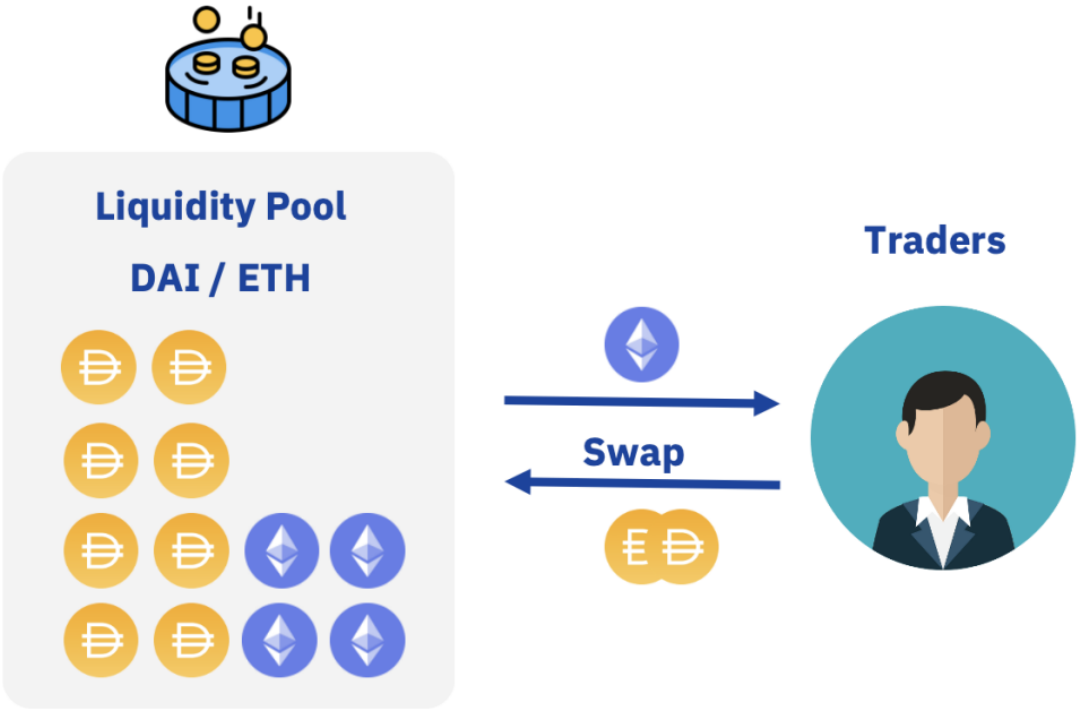


Decentralized exchange market

- Constant function market maker
  - Uniswap, Balancer…
- Combinatorial swap problem = range update query

Constant function market maker

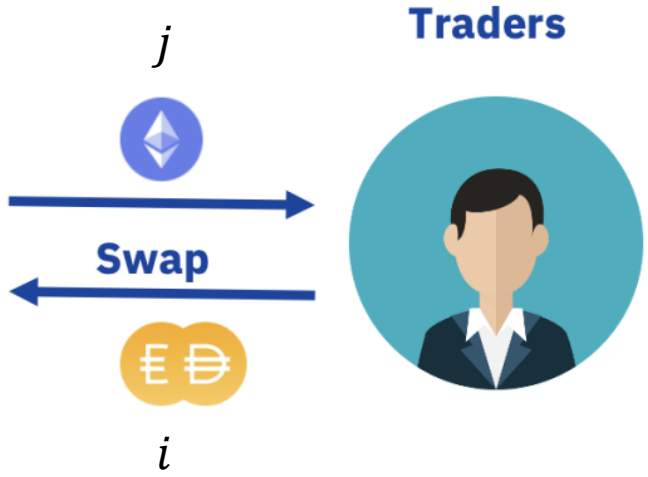
- AMM on states of n assets $w \in \mathbb{R}^n$ with ϕ on $\mathbb{R}^n$
- A trade $\Delta \in \mathbb{R}^n$ is valid if
$$\phi(w + \Delta) = \phi(w)$$
- E.g. Uniswap (v1/v2) $\phi(w) = \prod_i w_i$ (constant product market maker)



Swap trade

- Given $w \in \mathbb{R}^n$ and constant product market maker ϕ on $\mathbb{R}^n$, a swap trade $\Delta = s1_i - 1_j \in \mathbb{R}^n$
- A valid s satisfies

$$(w_i + s) = \frac{w_i w_j}{(w_j - 1)}$$



Swap trade for baskets

- Given $w \in \mathbb{R}^n, E, E' \subseteq [n]$ and constant product market maker ϕ on $\mathbb{R}^n$, a swap trade for baskets $\Delta = s1_E - 1_{E'} \in \mathbb{R}^n$
- A valid s satisfies

$$\prod_{i \in E}(w_i + s) = \frac{\prod_{i \in E} w_i \prod_{j \in E'} w_j}{\prod_{j \in E'}(w_i - 1)}$$

- The above can be reduced to RQRU with * query and + update

Summary

- AMM for combinatorial securities = range query problem
  - Sublinear Time LMSR for bounded VC dimension securities
  - Generalize: quadratic scoring rule and 3/2-power scoring rule
  - Incorporate multi-resolution market designs into a partition-tree scheme.
  - Combinatorial Swap in DeFi
- Future work
  - Better scoring rules for AMM
  - Approximation and dynamic data structure
  - Multi-resolution
  - CFMM