Packing Compact Subgraphs with Applications to Districting

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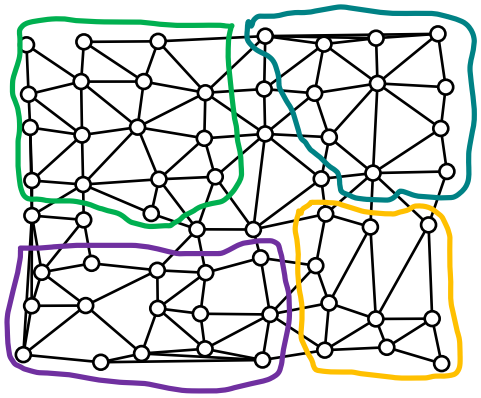
Packing Compact Subgraphs

- Given a graph $G = (V, E)$ with vertex weights $\vec{w} = (w_{obj}, w_1, \dots)$
  - District S: a connected subgraph with compactness and composition constraints.
  - Maximize the total weight of vertex-disjoint districts $\mathcal{S}$, $\sum_{S \in \mathcal{S}} w_{obj}(S)$.



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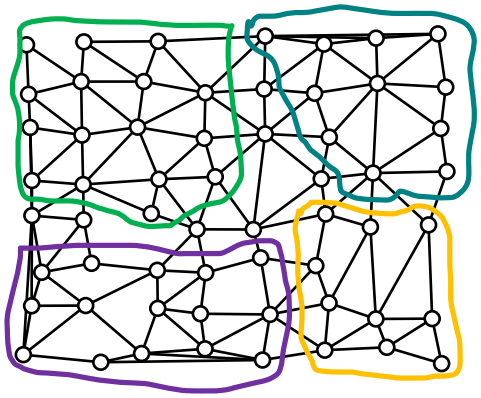
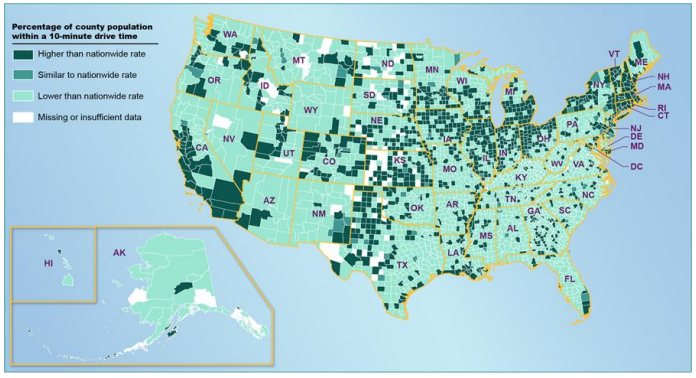


Facility allocation

Service regions: public schools, healthcare systems, police and emergency response units

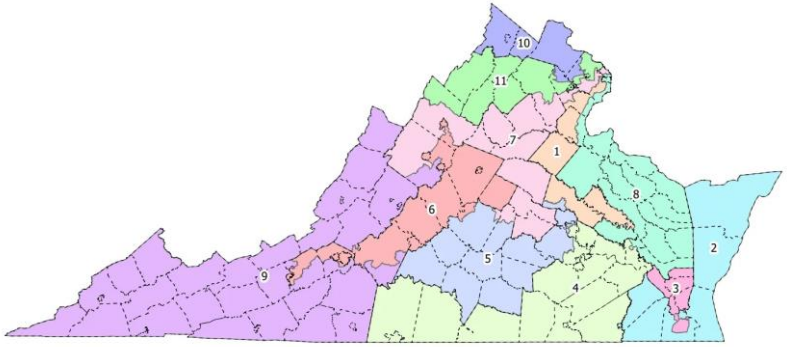
- Contiguity and compactness:
  - transportation cost, response time
- Composition: **minimum-threshold**
  - serve enough residents to justify a facility



Source: GAO analysis of Fiscal Year 2022 Public Libraries Survey and U.S. Census tracts. | GAO-26-107262

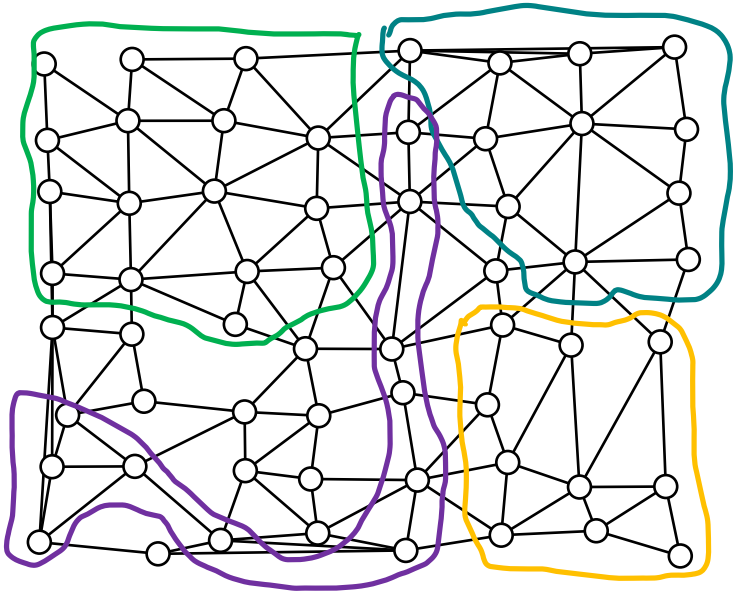
Political districting

- Contiguity and compactness:
  - Gerrymandering
  - Transportation cost
- Composition: **balanced** representation
  - Fair representation, Electoral competitiveness



Packing Compact Subgraphs

- Given a graph $G = (V, E)$ with vertex weights $\vec{w}$, find a collection $\mathcal{S}$ of connected subgraph $S \in \mathcal{S}$ with
  - Compactness: radius bounded by k
    - strong/weak, diameter, star (radius 1)
  - maximize $\sum_{S \in \mathcal{S}} w_{obj}(S)$



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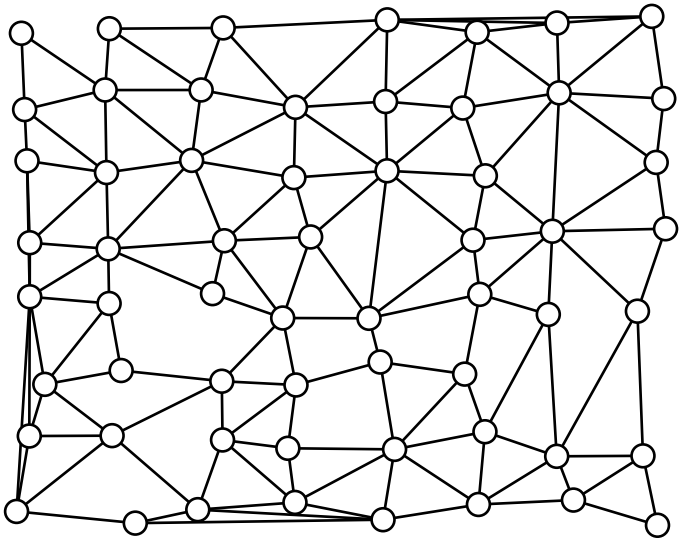

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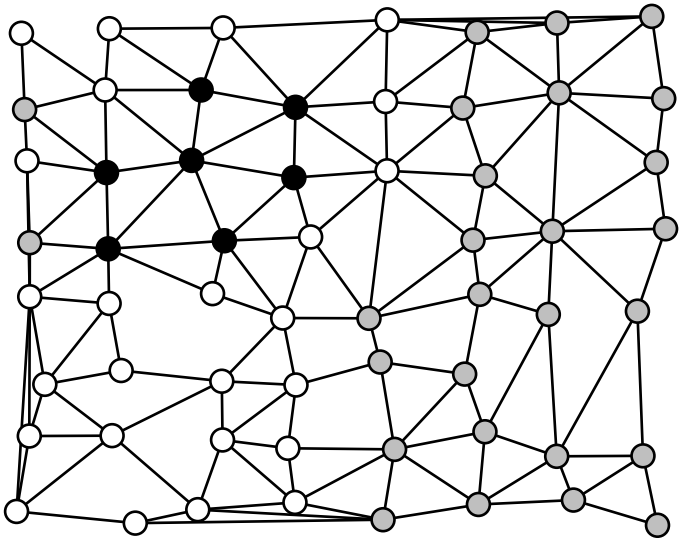

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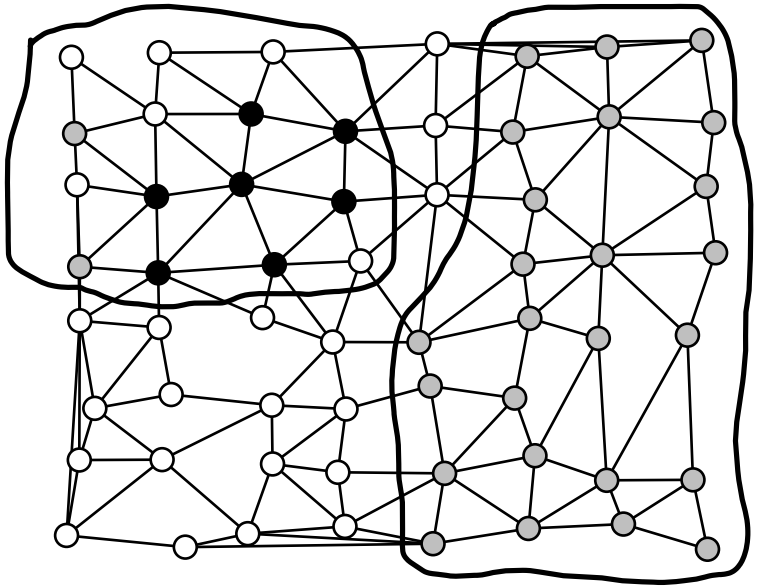

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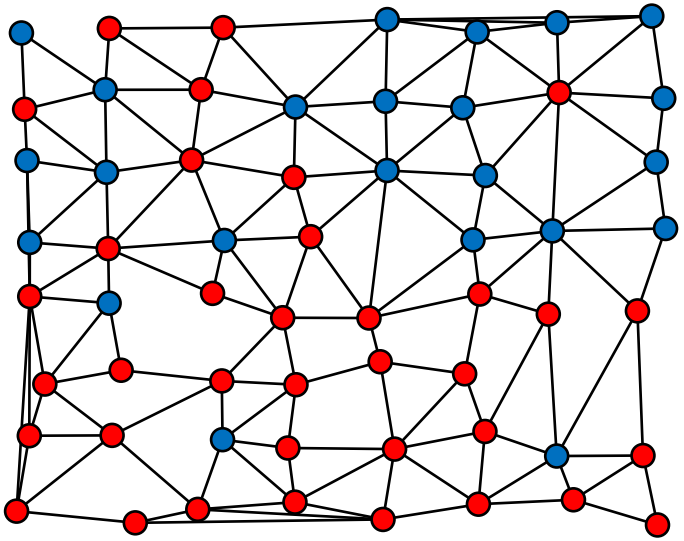
  - Composition
    - B-threshold $w_1(S) \geq B$
    - c-balanced $\frac{1}{c-1} \leq \frac{w_1(S)}{w_2(S)} \leq c - 1$
  - maximize $\sum_{S \in \mathcal{S}} w_{obj}(S)$



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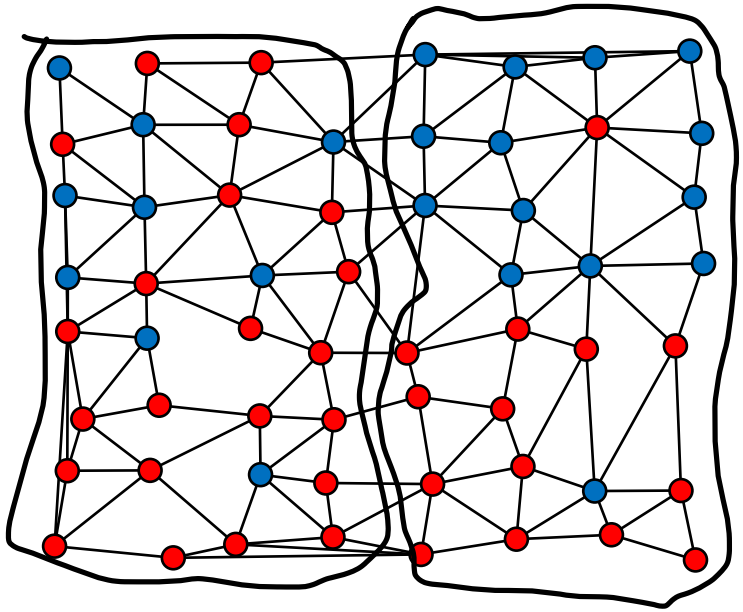

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    - B-threshold $w_1(S) \geq B$
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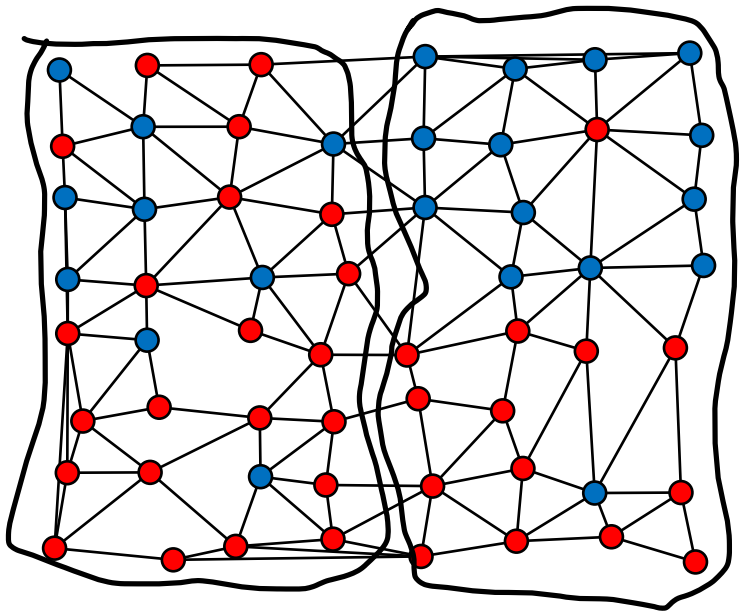
    - relaxed balanced or threshold
  - maximize $\sum_{S \in \mathcal{S}} w_{obj}(S)$



Main Results

- Efficient algorithms across several graph classes

Graph Type	District Type	Result
Planar	star (strong radius-1)	$O(\log n)$-approx [DGHY25]
	strong radius-k	$O(1)$-**approx**
	weak/strong radius-k	$(1+\epsilon)$-**approx,** δ-**relaxation**
Apex-Minor-Free	**strong radius-k**	$O(1)$-**approx**
	weak/strong radius-k	$(1+\epsilon)$-**approx,** δ-**relaxation**
H-Minor-Free	star (strong radius-1)	$O(h^2 \log n)$-approx, $h = \lvert H \rvert$ [DGHY25]
	star (strong radius-1)	$O(1)$-**approx**
Bounded Expansion	**star (strong radius-1)**	$O(1)$-**approx**

- Highlights:
  - Correlation ratio: a new LP-rounding bound for packing problems
  - $O(\log n) \to O(1)$ for planar graphs (and apex minor free)
  - 3-color-balancedness is NP-hard for any approximation ratio.

Algorithm idea: LP+rounding

- Formulate a packing LP over all feasible districts

$$\begin{aligned} \max \quad & \sum_S w(S) x_S \\ \text{s.t.} \quad & \sum_{S \ni v} x_S \leq 1 \qquad \forall v \in V, \\ & 0 \leq x_S \leq 1 \qquad \forall S. \end{aligned}$$

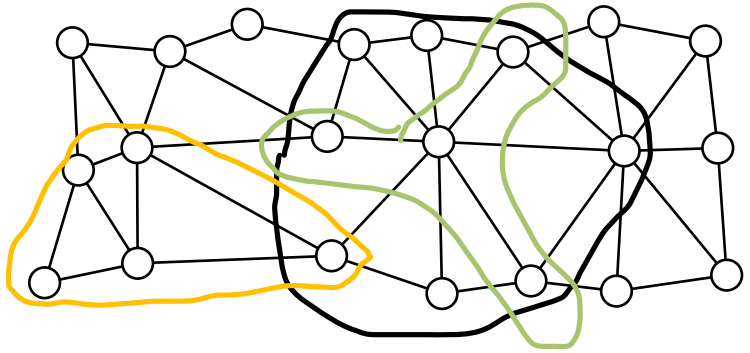
- Challenges
  - Rounding for disjoint districts: correlation ratio!
  - Exponentially many districts x_S: separation oracle

Correlation Ratio τ

- There is a rounding algorithm achieves $O(\tau)$-approximation if

$$\sum_{A \cap B \neq \emptyset} x_A x_B \leq \tau \sum_{A} x_A$$

  - LHS measures the total fractional mass of conflicting district pairs.
  - Correlation ratio τ can be bounded by maximum degree (too weak)



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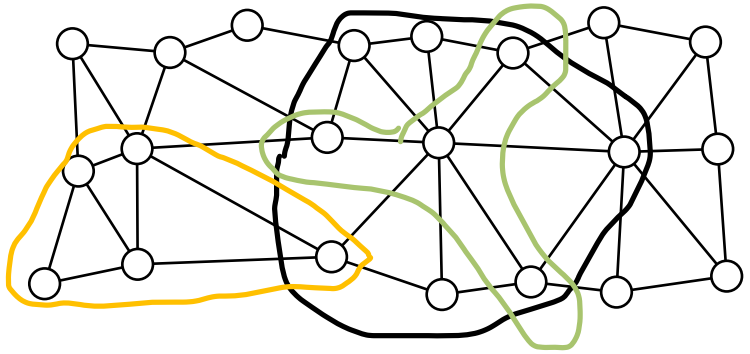
  - LHS measures the total fractional mass of conflicting district pairs.
  - Given a constant strong radius bound k for districts, we prove
    - General graphs: $\tau = O(\sqrt{n})$
    - H-minor-free graphs (including planar graphs): $\tau = O(1)$.
    - Bounded-expansion graphs: $\tau = O(1)$.



(1+ε)-approx via Baker's technique

- Given planar and apex-minor-free graphs,
  - Decompose by BFS layers and delete every t-th layer-> bounded treewidth on each components
  - Run the DP on the tree decomposition
  - Pigeonhole: best of the t offsets loses only an $O\left(\frac{1}{t}\right)$ fraction of OPT

Discussion

- Takeaways
  - Compact subgraph packing: packing LP + correlation-ratio rounding
  - The correlation ratio τ: $O(\sqrt{n})$general and $O(1)$ on planar/minor-free/bounded-expansion
  - Two composition modes: threshold and balancedness
- Open questions
  - Correlation Ratio vs. Integrality gap?
  - PTAS for exact composition constraints?
  - Other composition conditions
  - Covering or partition variants