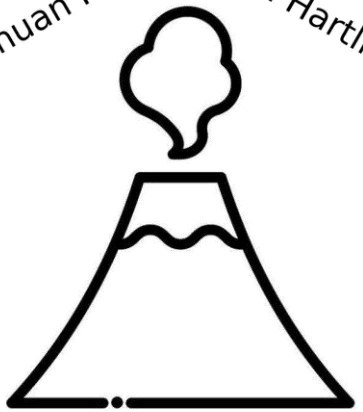
Northwestern University

George Mason University Department of Computer Science

Algorithmic Robust Forecast Aggregation



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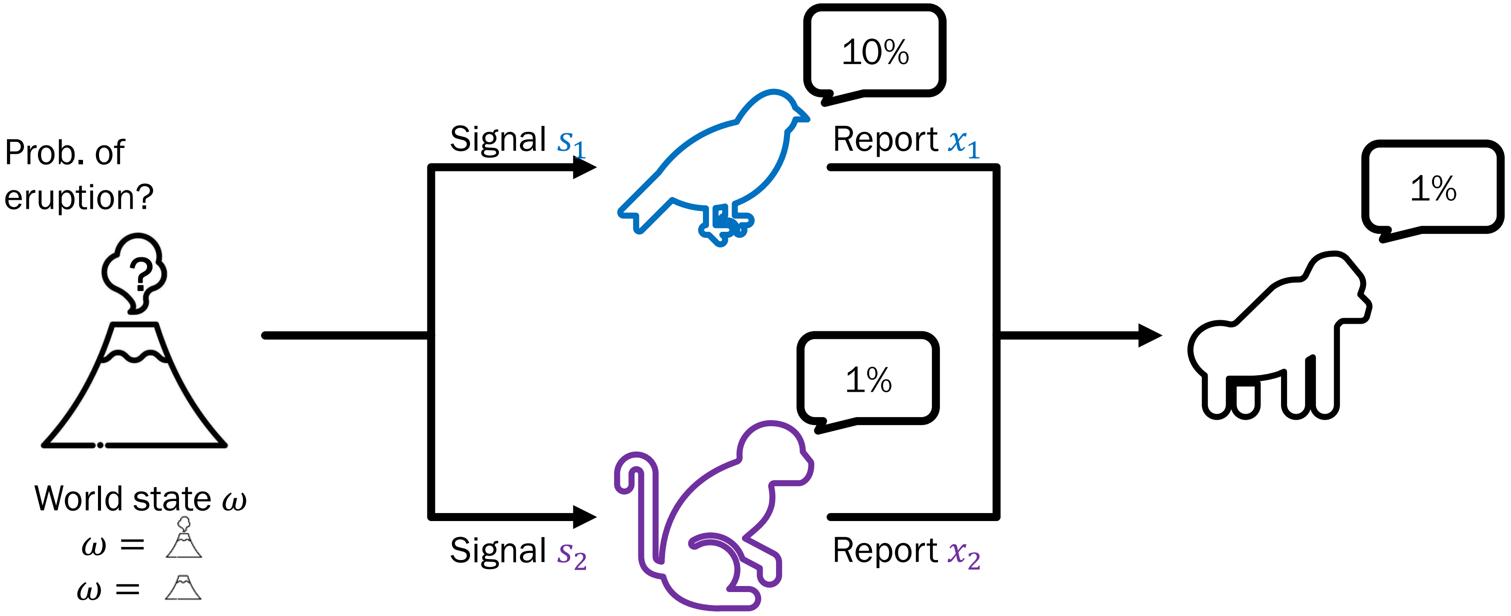
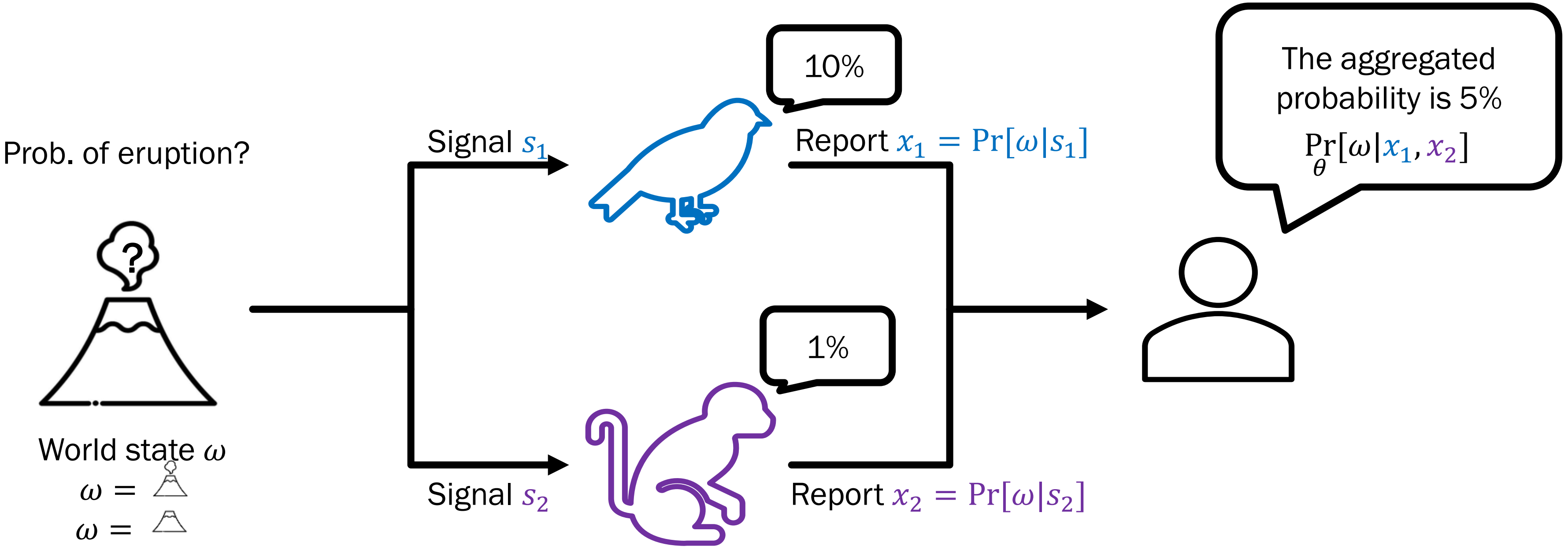
Center on Frontiers of Computing Studies, Peking University

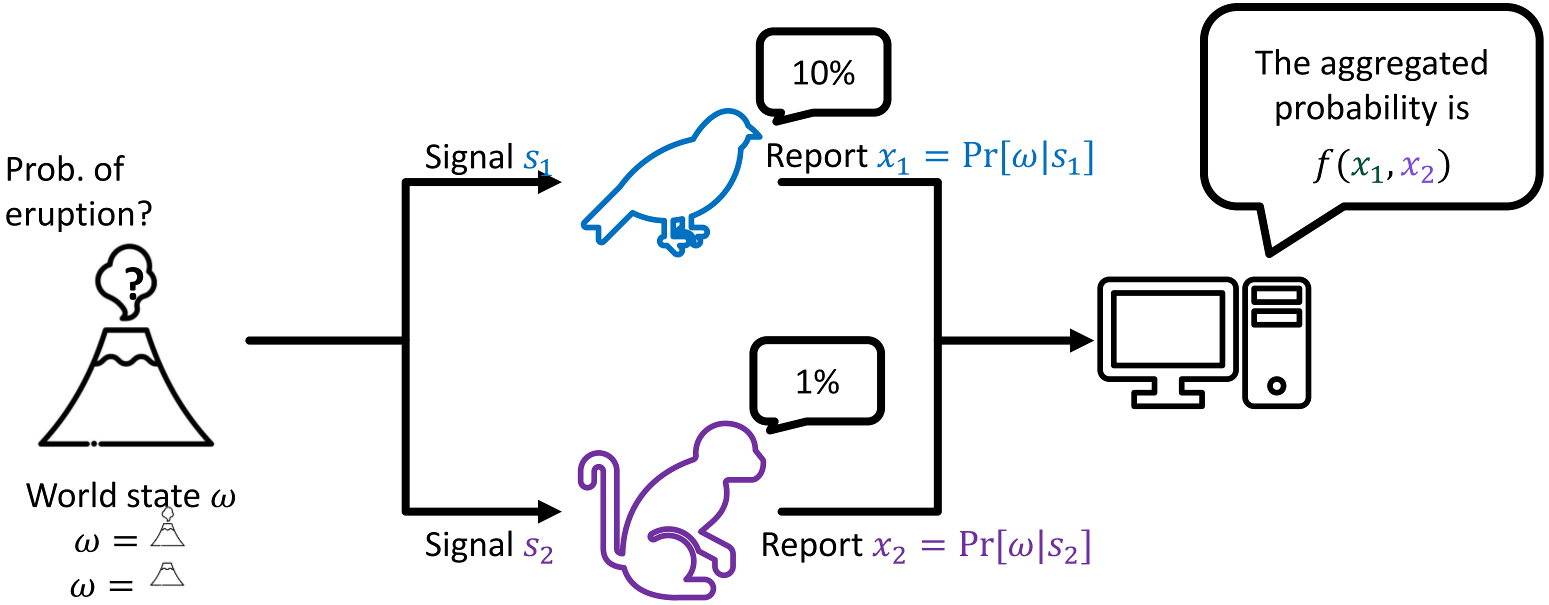
Long time ago…



When human master the Bayes rule…

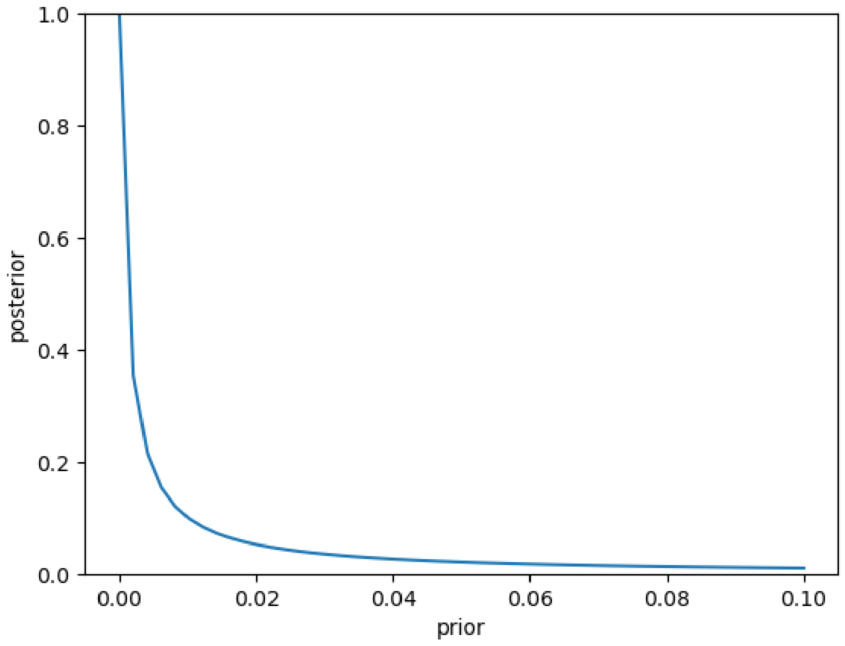


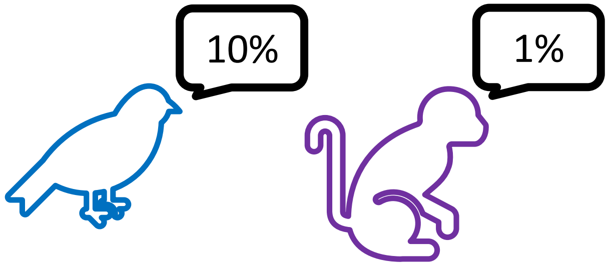
When we do not know joint distribution θ...



Prior matters

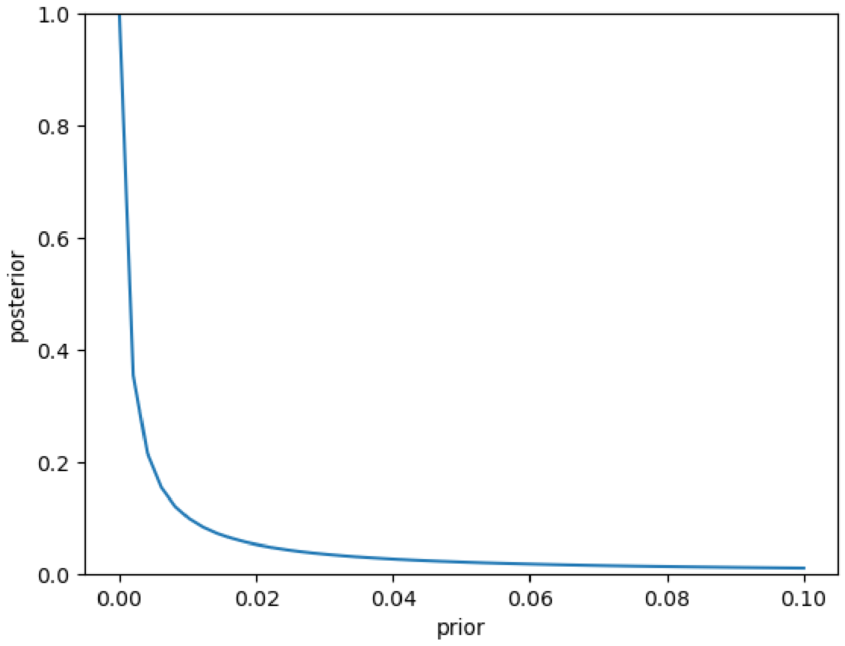
- If the prior probability of eruption μ
  - Given prediction x_1 and x_2, the posterior is

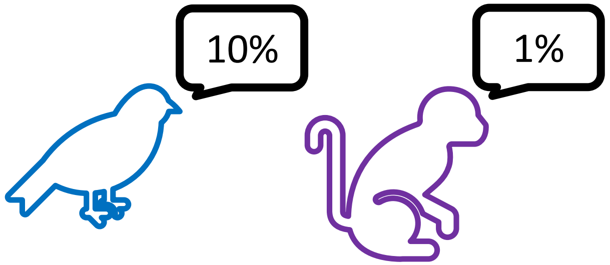




Prior Matters

- If the prior probability of eruption μ
  - Given prediction x_1 and x_2, the posterior is
$$\frac{(1-\mu)x_1x_2}{(1-\mu)x_1x_2+\mu(1-x_1)(1-x_2)}$$
  - 5% => safe
  - 0.000001% => **far less than** the forecasts => Dangerous!





Aggregating multiple forecasts

- Weather forecast
- Elections
- Investments
- Medical Prognosis

Outline

- Robust forecast aggregation problem
  - An ignorant aggregator
    - does not know the information structure
    - aims to minimize the regret in the worst information structures.
- Theoretical results
  - FPTAS based on online learning method
  - Two challenges
- Numerical results

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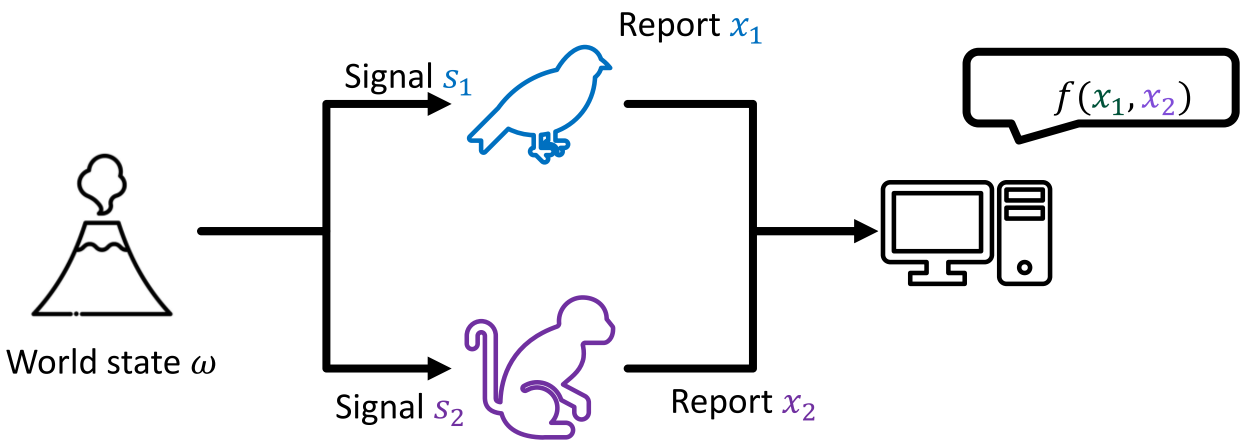
Robust Forecast Aggregation Problem [ABS18]

- Inputs
  - Binary state of nature $\Omega = \{0,1\}$
  - Two forecasters $\{1,2\}$ with signal sets S_1, S_2.
  - Information structure
    $$\theta \in \Delta(\Omega \times S_1 \times S_2)$$
  - Each forecaster reports the posterior
    $$x_i = \Pr_\theta[\omega = 1 | s_i]$$



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  - Each forecaster reports the posterior
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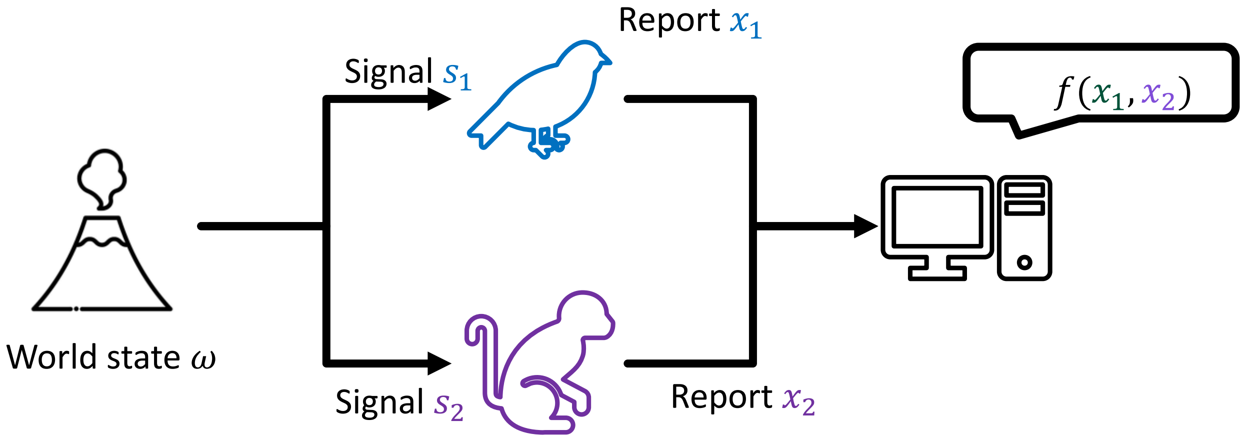
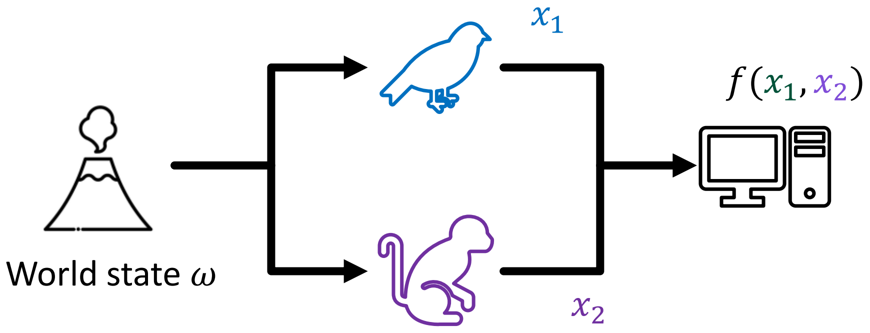
- Output a forecast $f(x_1, x_2)$ where $f: [0,1]^2 \to [0,1]$ is called an aggregation scheme



Measure aggregation schemes

The forecast should be accurate and robust

- Accurate:
  - Squared loss function $l(f,\theta) = \mathrm{E}_\theta[(f - w)^2]$,
  - Regret against Bayesian aggregator with θ
    $R(f,\theta) = l(f,\theta) - l(opt_\theta,\theta)$.



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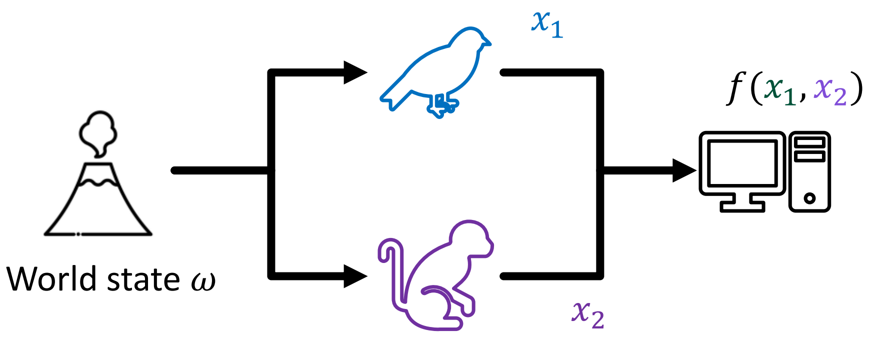
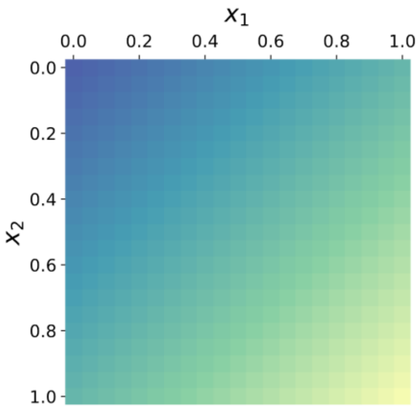
- Robust: a set of information structures Θ the worst-case performance is small

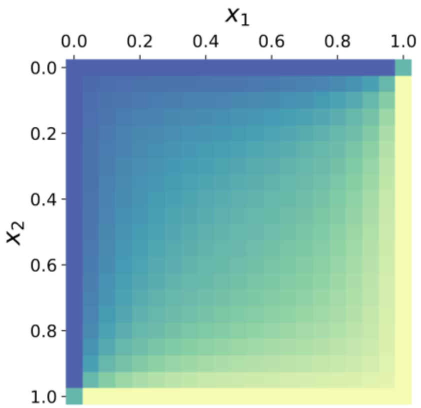
$$\inf_{f\in\mathcal{F}} \sup_{\theta\in\Theta} R(f,\theta)$$

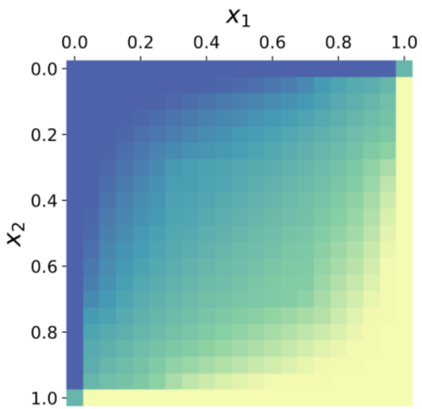


Possible Approaches

- Common function, e.g., average
- Delicate designed aggregators [ABS, PNAS 2018]
- An algorithmic and systematic method?







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FPTAS for optimal forecasts aggregation

There is an efficient algorithm outputting an ϵ-optimal aggregator for $\inf_{f \in \mathcal{F}} \sup_{\theta \in \Theta} R(f, \theta)$ if one of the following holds

1. Θ is finite
2. Θ consists of conditional independent information structures and $\mathcal{F}$ is the collection of L-Lipschitz aggregators

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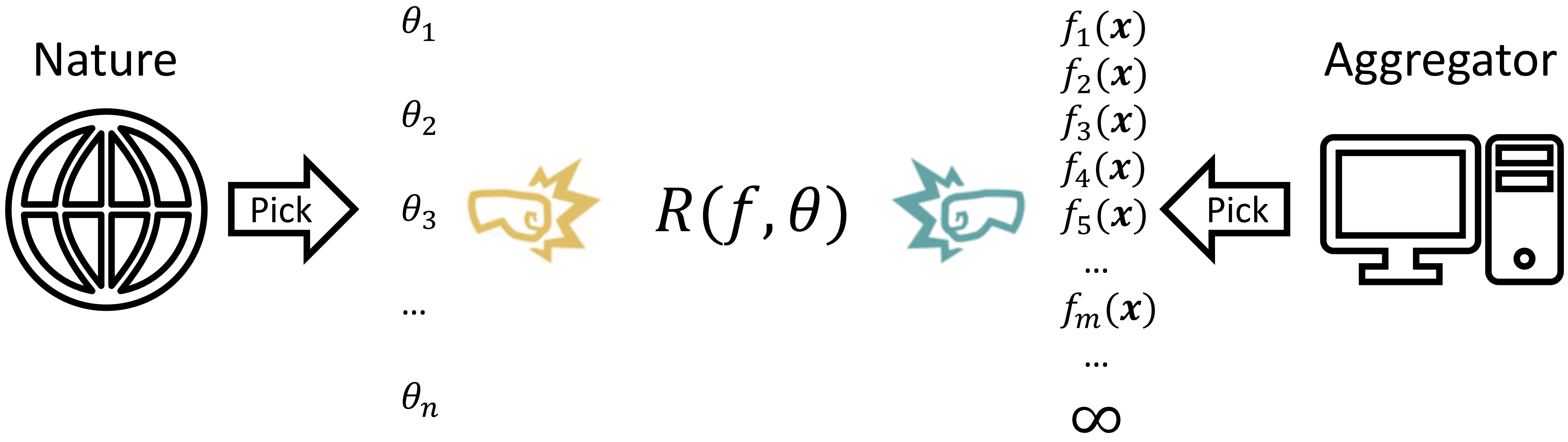
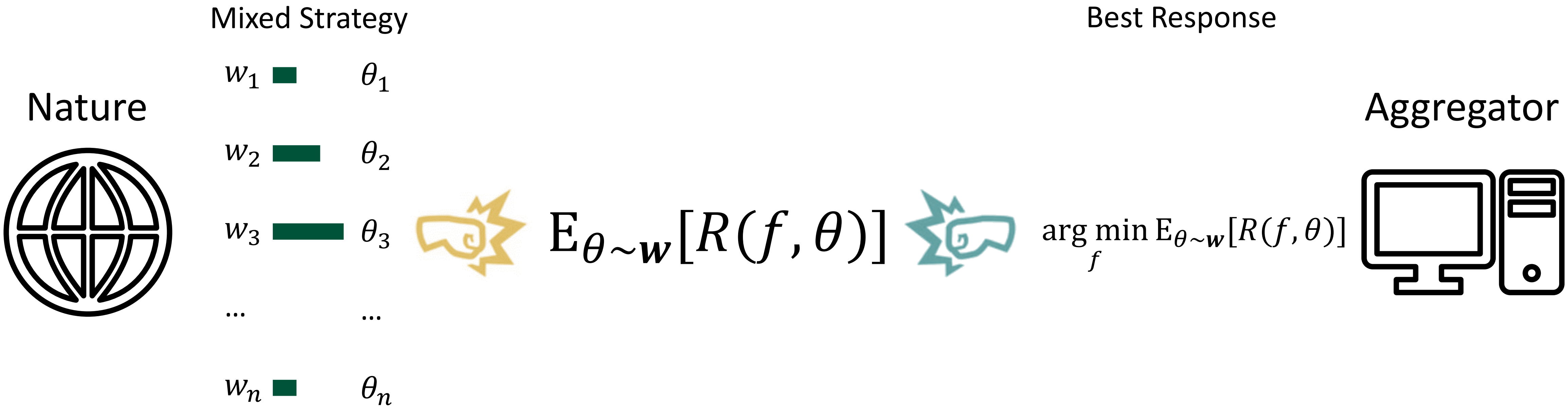
Algorithm: equilibrium computation

$\inf_{f \in \mathcal{F}} \sup_{\theta \in \Theta} R(f, \theta)$ as a zero-sum game



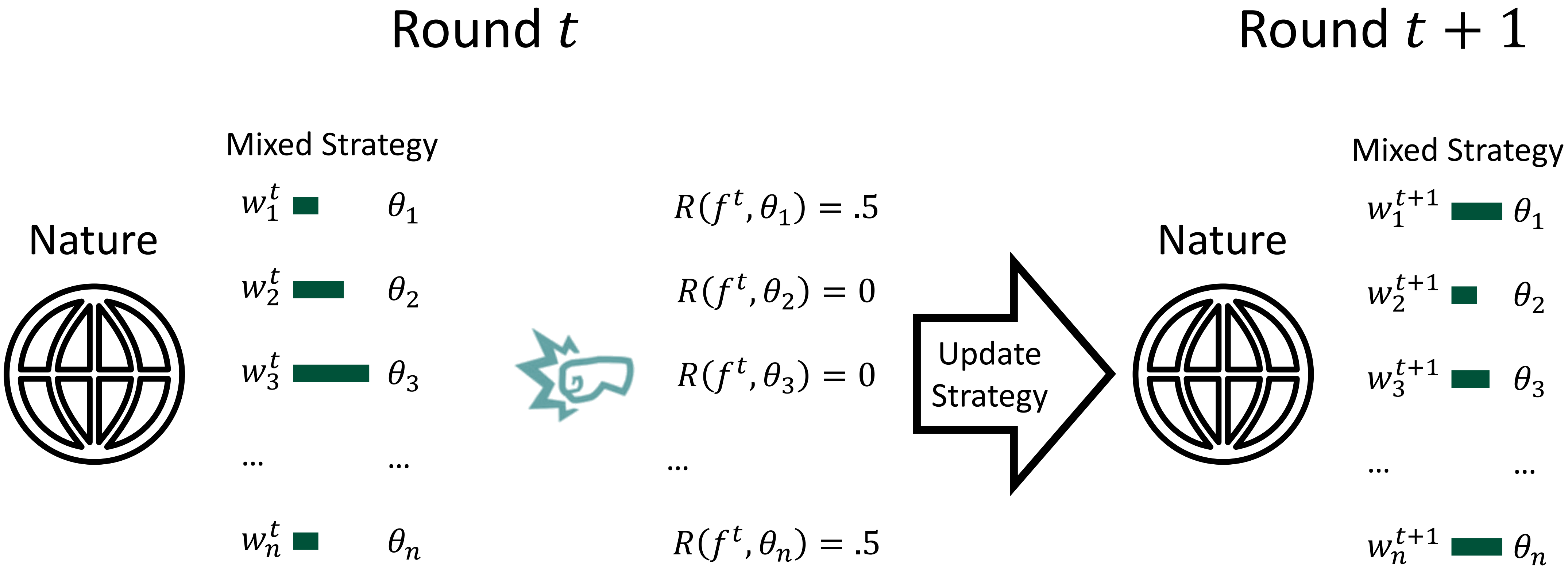
Online Learning: Efficient Best Response

No-regret vs Best response



Online Learning: Learn Mixed Strategy

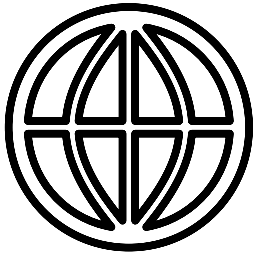
No-regret vs Best response



Online Learning: Optimal Solution

Nature

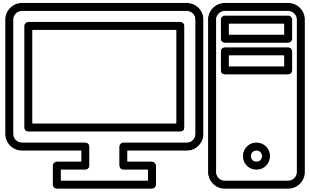
Aggregator



$$w^* = \frac{\sum_{t=1}^{T} w^t}{T}$$

$$\mathrm{E}_{\theta \sim w^*}[R(f^*, \theta)]$$

$$f^* = \frac{\sum_{t=1}^{T} f^t}{T}$$



$$\approx \inf_{f \in \mathcal{F}} \sup_{\theta \in \Theta} R(f, \theta)$$

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Two challenges

- The dimension of $\Theta \subset \Delta(\Omega \times S_1 \times S_2)$ is high.

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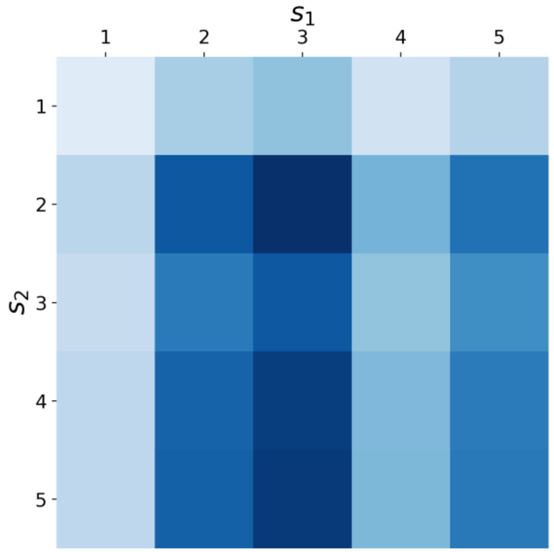
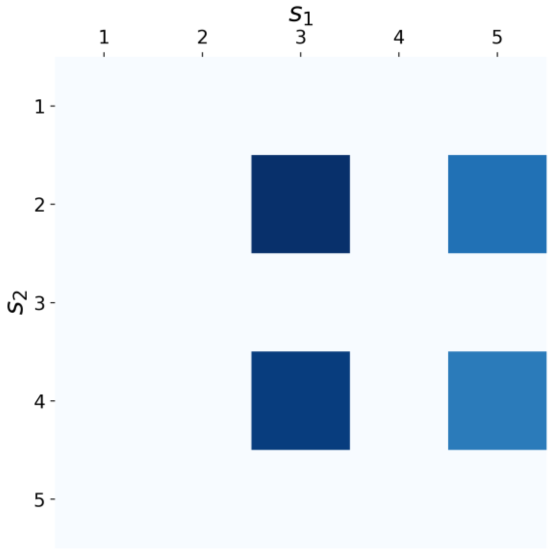
  - Dimension reduction

Multiple signals



= Linear Combination of

Binary signals



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Two challenges

- The dimension of $\Theta \subset \Delta(\Omega \times S_1 \times S_2)$ is high.
  - Dimension reduction
- Θ and $\mathcal{F}$ are continuous.
  - Discretize Θ to finite information structure Θ^{fin} and show $R(\Theta, \mathcal{F}) \approx R\left(\Theta^{fin}, \mathcal{F}\right)$

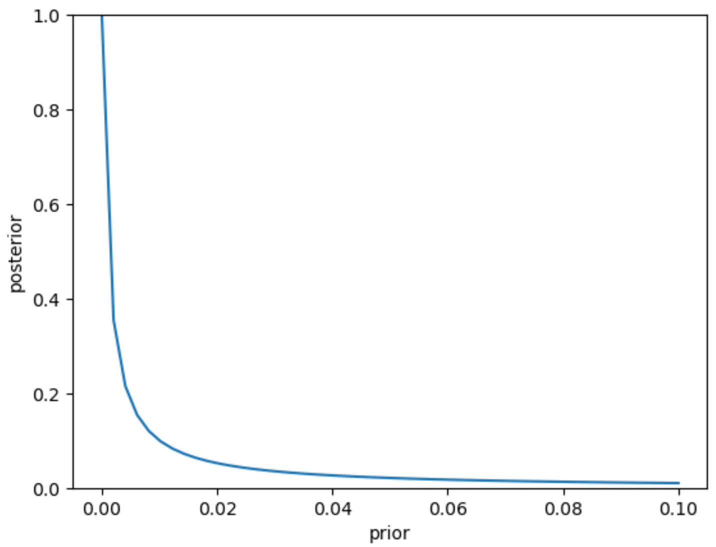
Control the Regret $R(\Theta, \mathcal{F}) \approx R\left(\Theta^{fin}, \mathcal{F}\right)$

- (ϵ, d)-Covering
  - For any $\theta \in \Theta$, we can find $\theta' \in \Theta'$ such that $d(\theta, \theta') \leq \epsilon$.
  - Use the nearest information structure in the Euclidean space.
  - Prove they are close in the TVD(EMD) metric.
- Discretize in two steps:
  - Discrete reports (EMD)
  - Discrete prior (TVD)

Control the Regret $R(\Theta, \mathcal{F}) \approx R\left(\Theta^{fin}, \mathcal{F}\right)$

$$R(f^t, \theta) = l(f^t, \theta) - l(\mathrm{opt}_\theta, \theta)$$

- Bayesian aggregator opt_θ is not smooth
  - – Find a smooth one to approximate opt_θ
  - – Trimming the non-smooth area.
  - – Extend the smooth regret to the full space.



- Best-response f^t is not smooth
  - – Restrict f^t to smooth function.
  - – Calculate the best smooth response by Ellipsoid method.

Two solutions

- Reduce dimension
  - The dimension of $\Theta \subset \Delta(\Omega \times S_1 \times S_2)$ is high.
- Discretize Θ and run no regret+best response
  - Calculate the best response f in $\mathcal{F}$.
  - Control the regret $R(\Theta, \mathcal{F}) \approx R\left(\Theta^{fin}, \mathcal{F}\right)$

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Our Results (Numerical)

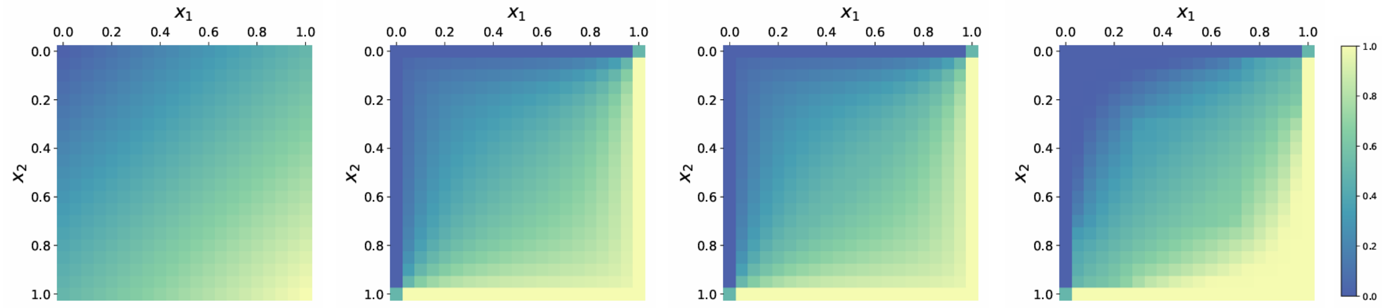
Aggregator	Formula	Regret
Simple average	$\frac{x_1+x_2}{2}$	0.0625
Average prior	$\frac{x_1x_2(1-\frac{x_1+x_2}{2})}{x_1x_2(1-\frac{x_1+x_2}{2})+(1-x_1)(1-x_2)\frac{x_1+x_2}{2}}$	0.0260
State-of-the-art	$\frac{x_1x_2(1-ep(x_1,x_2))}{x_1x_2(1-ep(x_1,x_2))+(1-x_1)(1-x_2)ep(x_1,x_2)}$	0.0250
Our	-	**0.0227**

Table 1: **Regret of different aggregators.**

Here $ep(x_1, x_2) = \begin{cases} 0.49x_1 + 0.49x_2, & if\ x_1 + x_2 \leq 1 \\ 0.49x_1 + 0.49x_2 + 0.02, & otherwise \end{cases}$

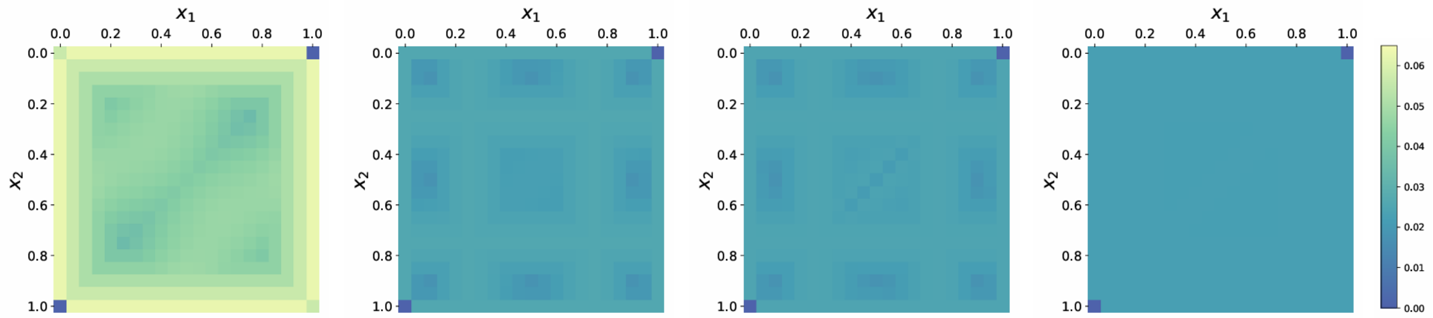
Our Results (Numerical)

- Extremization



(a) **Simple averaging** (b) **Average prior** (c) **State-of-the-art** (d) **Our aggregator**

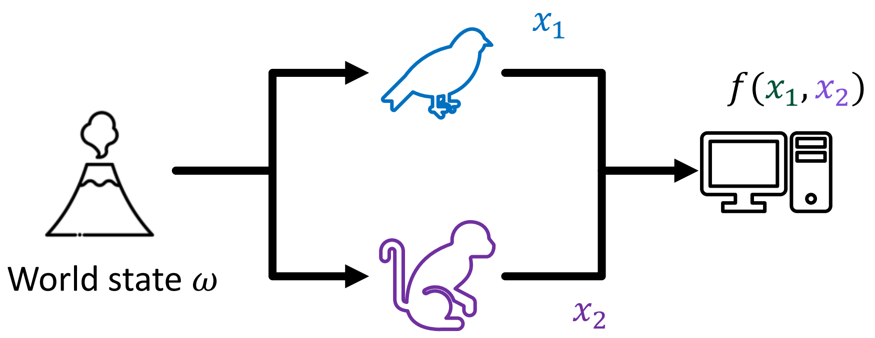
- Uniform loss



(a) **Simple averaging** (b) **Average prior** (c) **State-of-the-art** (d) **Our aggregator**

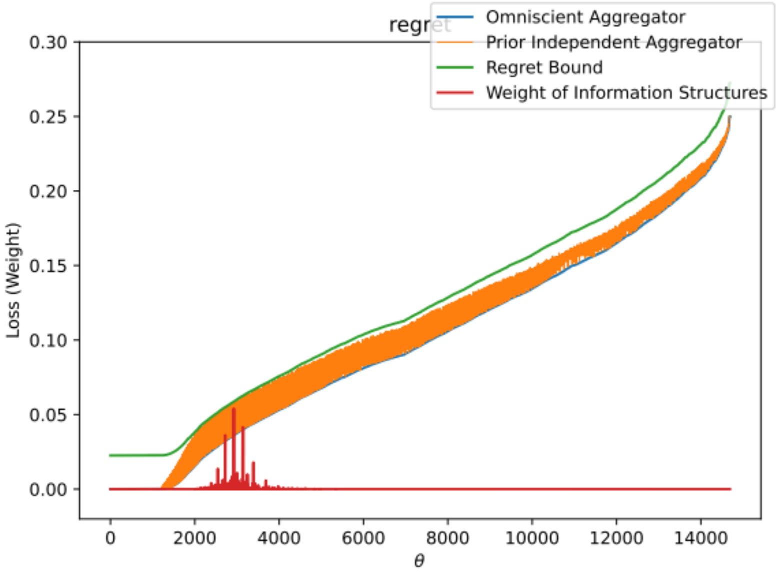
Conclusion and Future Work

- Conclusion
  - An algorithmic framework for robust information aggregation
  - Automatic design of aggregators, even in scenarios with complex report formats
- Future work using partial knowledge Θ
  - Aggregation
    - Higher order reports
    - Decision
  - Elicitation
    - Proper scoring rule

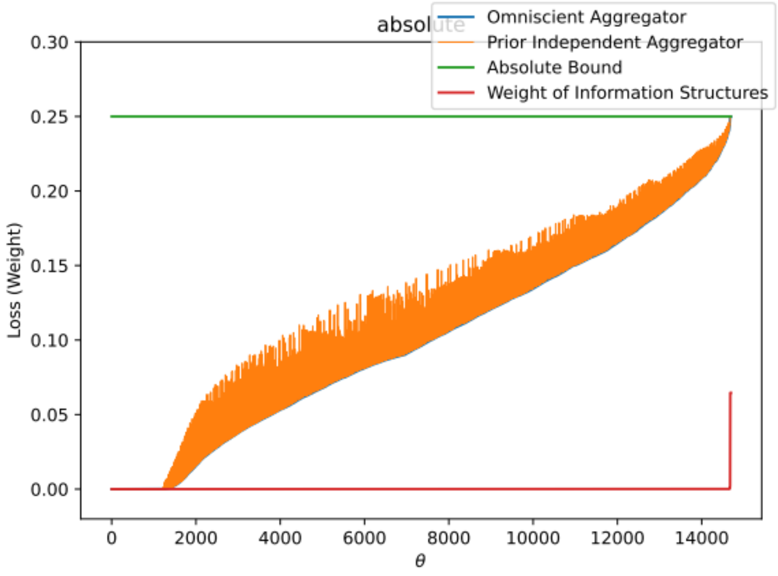


Additive vs. Ratio vs. Absolute

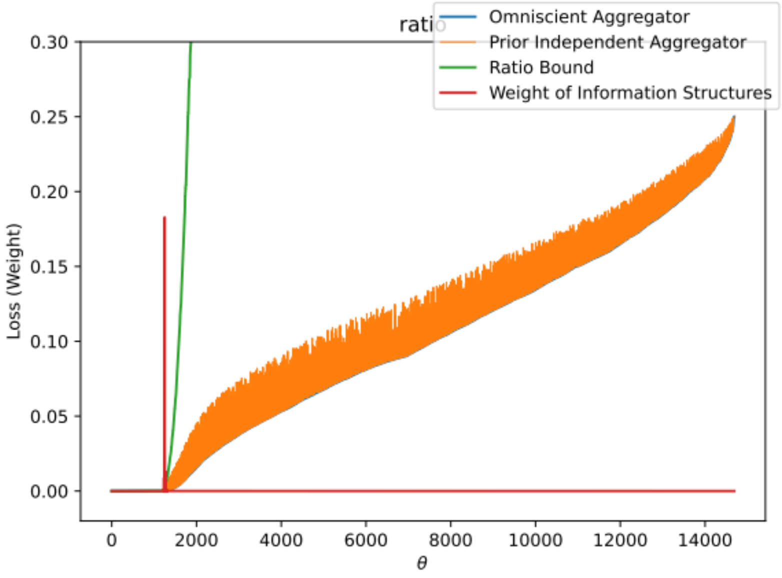
- $l(f^t, \omega) - l(\text{opt}_\theta, \omega)$



(a) **Additive**



(b) **Absolute**



(c) **Ratio**