Perfect Secrecy

Definition

An encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ is perfectly secret if for every probability distribution M over $\mathcal{M}$, every message $m \in \mathcal{M}$, and every ciphertext $c \in \mathcal{C}$ for which $\Pr[C = c] > 0$: $\Pr[M = m \mid C = c] = \Pr[M = m]$.

Here C is the random variable that results from sampling $m \leftarrow M$, $k \leftarrow \mathsf{Gen}$, and outputting $\mathsf{Enc}(k, m)$.

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Consider the Caesar Cipher, with messages that are at most 1 character long.
Consider the following message distribution, $\mathcal{M}$:
$Pr[m = \text{``}a\text{''}] = .7$ and $Pr[m = \text{``}z\text{''}] = .3$

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$$\Pr[m = \text{``}a\text{''} \mid C = b] = \frac{\Pr[C = b \mid m = \text{``}a\text{''}]\Pr[m = \text{``}a\text{''}]}{\Pr[C = b]}$$

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&= \frac{\Pr[k = 1]\Pr[m = \text{“}a\text{''}]}{\Pr[C = b]}
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&= \frac{\Pr[k = 1]\Pr[m = \text{“}a\text{''}]}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b]}
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&= \frac{\Pr[k = 1] \Pr[m = \text{``}a\text{''}]}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\sum_{m' \in \mathcal{M}} \Pr[C = b \mid m = m'] \Pr[m = m']}
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&= \frac{\Pr[k = 1]\Pr[m = \text{“}a\text{''}]}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\sum_{m' \in \mathcal{M}} \Pr[C = b \mid m = m']\Pr[m = m']} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b \mid m = \text{“}a\text{''}](.7) + \Pr[C = b \mid m = \text{“}z\text{''}](.3)}
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&= \frac{\Pr[k = 1] \Pr[m = \text{``}a\text{''}]}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b]} \\
&= \frac{(\frac{1}{26})(.7)}{\sum_{m' \in \mathcal{M}} \Pr[C = b \mid m = m'] \Pr[m = m']} \\
&= \frac{(\frac{1}{26})(.7)}{\Pr[C = b \mid m = \text{``}a\text{''}](.7) + \Pr[C = b \mid m = \text{``}z\text{''}](.3)} \\
&= \frac{(\frac{1}{26})(.7)}{\frac{1}{26}(.7) + \frac{1}{26}(.3)} = \frac{.7}{.7 + .3} = .7
\end{aligned}$$

One Time Pad

Message space $\mathcal{M}$: $\{\mathrm{a}, \ldots, \mathrm{z}\}^\ell$
Keyspace $\mathcal{K}$: $\{0, \ldots, 25\}^\ell$
Ciphertext space $\mathcal{C}$: $\{\mathrm{a}, \ldots, \mathrm{z}\}^\ell$

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Gen : $k = k_1 \ldots k_\ell \leftarrow \mathcal{K}$
Enc(k, m) : $c_i = m_i + k_i \mod 26$
Dec(k, c) : $m_i = c_i - k_i \mod 26$

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Consider the following message distribution, M:

$$\Pr_{m \leftarrow M}[m = \text{“kim”}] = .5$$
$$\Pr_{m \leftarrow M}[m = \text{“ann”}] = .2$$
$$\Pr_{m \leftarrow M}[m = \text{“boo”}] = .3$$

Suppose the adversary sees ciphertext $c =$ “DQQ”.

Prove that:

$$\Pr[m = \text{“kim”} \mid c = \text{“DQQ”}] = \Pr[m = \text{“kim”}]$$
$$\Pr[m = \text{“ann”} \mid c = \text{“DQQ”}] = \Pr[m = \text{“ann”}]$$
$$\Pr[m = \text{“boo”} \mid c = \text{“DQQ”}] = \Pr[m = \text{“boo”}]$$

One Time Pad

Usually we use Binary strings instead of the alphabet $\{a, \ldots, z\}$.
Message space $\mathcal{M}$: $\{0,1\}^\ell$
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$\text{Gen} : k = k_1 \ldots k_\ell \leftarrow \mathcal{K}$
$\text{Enc}(k, m) : c_i = m_i + k_i \mod 2$
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Example, for $\ell = 3$:
$k = 101$
$\text{Enc}(k, 111) = 010$

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$$
\begin{aligned}
\Pr[c = 010] = {} & \Pr[c = 010 \mid m = 000] \Pr[m = 000] \\
& + \Pr[c = 010 \mid m = 001] \Pr[m = 001] \\
& + \Pr[c = 010 \mid m = 010] \Pr[m = 010] \\
& + \Pr[c = 010 \mid m = 011] \Pr[m = 011] \\
& + \Pr[c = 010 \mid m = 100] \Pr[m = 100] \\
& + \Pr[c = 010 \mid m = 101] \Pr[m = 101] \\
& + \Pr[c = 010 \mid m = 110] \Pr[m = 110] \\
& + \Pr[c = 010 \mid m = 111] \Pr[m = 111]
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$$\begin{aligned}
\Pr[c = 010] = {} & \Pr[c = 010 \mid m = 000] \Pr[m = 000] = (1/8) \Pr[m = 000] \\
& + \Pr[c = 010 \mid m = 001] \Pr[m = 001] = (1/8) \Pr[m = 001] \\
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& + \Pr[c = 010 \mid m = 011] \Pr[m = 011] = (1/8) \Pr[m = 011] \\
& + \Pr[c = 010 \mid m = 100] \Pr[m = 100] = (1/8) \Pr[m = 100] \\
& + \Pr[c = 010 \mid m = 101] \Pr[m = 101] = (1/8) \Pr[m = 101] \\
& + \Pr[c = 010 \mid m = 110] \Pr[m = 110] = (1/8) \Pr[m = 110] \\
& + \Pr[c = 010 \mid m = 111] \Pr[m = 111] = (1/8) \Pr[m = 111]
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& + \Pr[c = 010 \mid m = 101]\Pr[m = 101] = (1/8)\Pr[m = 101] \\
& + \Pr[c = 010 \mid m = 110]\Pr[m = 110] = (1/8)\Pr[m = 110] \\
& + \Pr[c = 010 \mid m = 111]\Pr[m = 111] = (1/8)\Pr[m = 111] \\
= {} & 1/8
\end{aligned}
$$

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$$\begin{aligned}\Pr[M = m \mid C = c] &= \frac{\Pr[C = c \mid M = m] \Pr[M = m]}{\Pr[C = c]} \\ &= \frac{\frac{1}{8} \Pr[M = m]}{\frac{1}{8}} \\ &= \Pr[M = m]\end{aligned}$$

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Definition 2

For every $m_1, m_2 \in \mathcal{M}$, and every $c \in \mathcal{C}$, $\Pr[\mathrm{Enc}(k, m_1) = c] = \Pr[Enc_k(m_2) = c]$.

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$$\Pr[M = m \mid C = c] = \Pr[C = c \mid M = m] \Pr[M = m] / \Pr[C = c]$$

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$$\begin{aligned}\Pr[M = m \mid C = c] &= \Pr[C = c \mid M = m] \Pr[M = m] / \Pr[C = c] \\ &= \frac{\Pr[C = c \mid M = m] \Pr[M = m]}{\sum_{m' \in \mathcal{M}} \Pr[C = c \mid M = m'] \Pr[M = m']}\end{aligned}$$

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&= \frac{\delta \Pr[M = m]}{\delta \sum_{m' \in \mathcal{M}} \Pr[M = m']} \\
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Suppose $(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ satisfy Definition 1.
$\forall m \in \mathcal{M}, \forall c \in \mathcal{C}, \forall M \text{ over } \mathcal{M}$,
$\Pr[M = m \mid C = c] = \Pr[M = m]$.

Perfect Secrecy, another way

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An encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ is perfectly secret if for every probability distribution M over $\mathcal{M}$, every message $m \in \mathcal{M}$, and every ciphertext $c \in \mathcal{C}$ for which $\Pr[\mathcal{C} = c] > 0$: $\Pr[M = m \mid C = c] = \Pr[M = m]$.

Definition 2

For every $m_1, m_2 \in \mathcal{M}$, and every $c \in \mathcal{C}$, $\Pr[\mathrm{Enc}(k, m_1) = c] = \Pr[Enc_k(m_2) = c]$.

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One Time Pad: Second Proof

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Keyspace $\mathcal{K}$: $\{0,1\}^\ell$
Ciphertext space $\mathcal{C}$: $\{0,1\}^\ell$

Gen : $k = k_1 \dots k_\ell \leftarrow \mathcal{K}$
$\mathrm{Enc}(k, m) : c_i = m_i + k_i \mod 2$
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Proof: $\forall m \in \mathcal{M}, \Pr[\text{Enc}(k, m) = c] = \Pr[k \oplus m = c] = \Pr[k = m \oplus c] = 2^{-\ell}$

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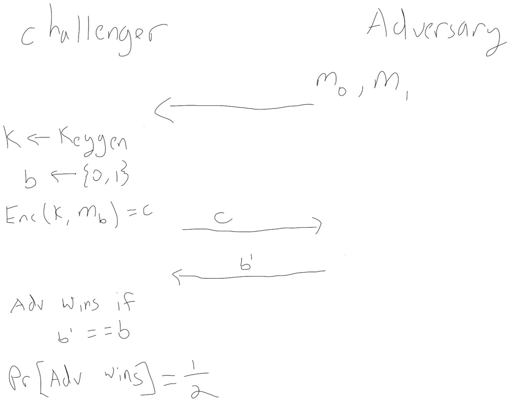

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Let M be the uniform distribution over $\mathcal{M}$.
$\Pr[M = m^*] = \frac{1}{|\mathcal{M}|}$.
$\Pr[M = m^* \mid C = c] = 0$

Reuse One Time Pad

What happens if we reuse the key in the OTP?

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This is called a *Known Plaintext Attack.* We'd like to protect against this.