PRGs

Intuition: A Pseudorandom Generator (PRG) takes a *small, uniformly random seed*, and stretches it into a longer string that is *not* uniformly random, but is indistinguishable from random.

PRGs

Definition (PRG)

Let ℓ be a polynomial, and let G be a deterministic polynomial-time algorithm such that for any n and any input $s \in \{0,1\}^n$, $G(s)$ is a string of length $\ell(n)$. We say that G is a pseudorandom generator if the following conditions hold:

1. Expansion: for every n it holds that $\ell(n) > n$.
2. Pseudorandomness: for any PPT algorithm $\mathcal{A}$, there is a negligible function negl(n) such that $\Pr[\mathsf{PrivK}^{\mathsf{prg}}_{\mathcal{A},\mathsf{G}}(n) = 1] \leq \frac{1}{2} + \mathsf{negl(n)}$

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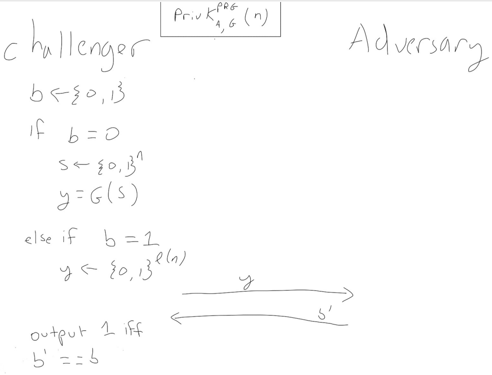

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G is *NOT* pseudorandom if: $\exists$ a PPT algorithm $\mathcal{A}$ and some polynomial $p(\cdot)$, s.t. $\Pr[\mathrm{PrivK}^{\mathrm{prg}}_{\mathcal{A},\mathrm{G}}(n) = 1] > \frac{1}{2} + \frac{1}{p(n)}$

An Insecure PRG

$G(s_1 \cdots s_n)$:
Let $s_{n+1} = \bigoplus_{i \in \{1,\ldots,n\}} s_i$
Output $s_1 \cdots s_{n+1}$

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Pseudorandom vs. Random

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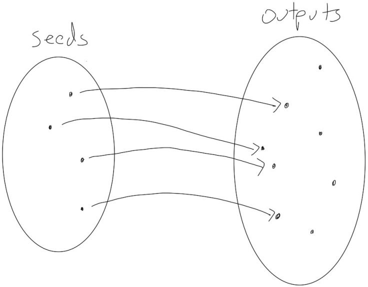

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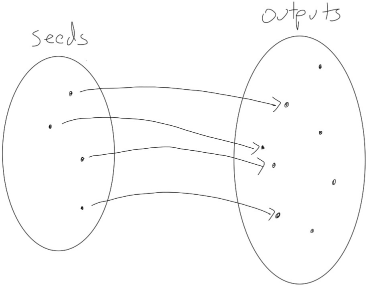

For input of length n:
How many different input seeds are there?
How many different outputs does G have (maximum)?
How many strings of length $2n$ are there?
If you choose $y \leftarrow \{0,1\}^{2n}$ (i.e. uniformly at random),
what is the probability that you choose an output of G?

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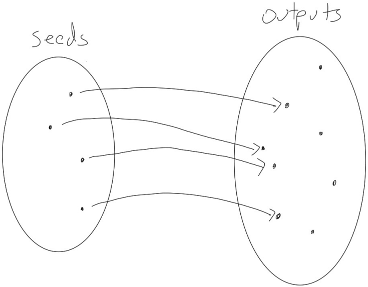

For input of length n:

How many different input seeds are there?	2^n
How many different outputs does G have (maximum)?	2^n
How many strings of length $2n$ are there?	2^{2n}

If you choose $y \leftarrow \{0,1\}^{2n}$ (i.e. uniformly at random),
what is the probability that you choose an output of G? $\frac{2^n}{2^{2n}} = 2^{n-2n} = 2^{-n}$

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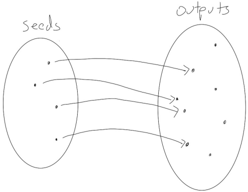

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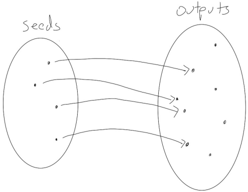

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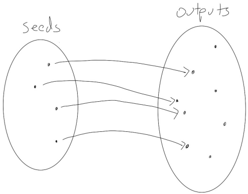

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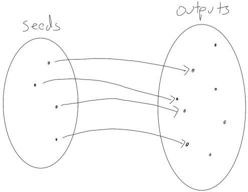

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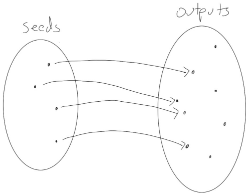

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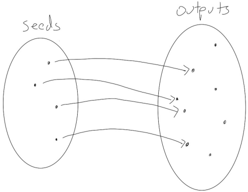

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