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Definition

A function $f : \{0,1\}^* \to \{0,1\}^*$ is *one-way* if the following two conditions hold:

- (Easy to compute:) There exists a polynomial-time algorithm M_f computing f; that is, $M_f(x) = f(x)$ for all x.
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Candidate owf: $f_{p,g}(x) = g^x \bmod p$

Hard-Core Predicates

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A function $\mathsf{hc} : \{0,1\}^* \to \{0,1\}$ is *a hard-core predicate of a function f* if hc can be computed in polynomial time, and for every probabilistic polynomial-time adversary $\mathcal{A}$ there is a negligible function negl such that

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Theorem

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Claim: $\exists p.p.t.\ \mathcal{A}$ s.t. $\Pr[\mathcal{A}(1^n, (f(x), r)) = \mathrm{gl}(x, r)] = 1$,
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Proof: On input $(1^n, y)$, $\mathcal{A}_r$ sends n different challenges to $\mathcal{A}$: $\{(1^n, (y, e^i))\}_{i=1}^{n}$, where e^i is the vector of length n, containing a 1 in location i, and 0 everywhere else.

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$$\bigoplus_{j=1}^{n} (x_j \wedge e_j^i) = x_i.$$

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Claim: $\exists p.p.t.\ \mathcal{A}$ s.t. $\Pr[\mathcal{A}(1^n, (f(x), r)) = \mathrm{gl}(x, r)] \geq \frac{3}{4} + \frac{1}{\mathrm{poly}(n)}$
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Note that $\mathrm{gl}(x, r) \oplus \mathrm{gl}(x, r \oplus e^i) = x_i$.

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Proof: see book.

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Thereom

Let f be a one-way permutation with hard-core predicate hc. Then $G(s) = f(s)||\text{hc}(s)$ is a PRG with expansion factor $\ell(n) = n + 1$.

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Key observation in the analysis: note that $f(x)$ is uniformly distributed, since x is uniform, and f is a permutation.

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Output it all.

PRF from PRG

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Let G be a PRG with expansion factor $\ell(n) = 2n$. Then there exists a fixed-length PRF $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$.

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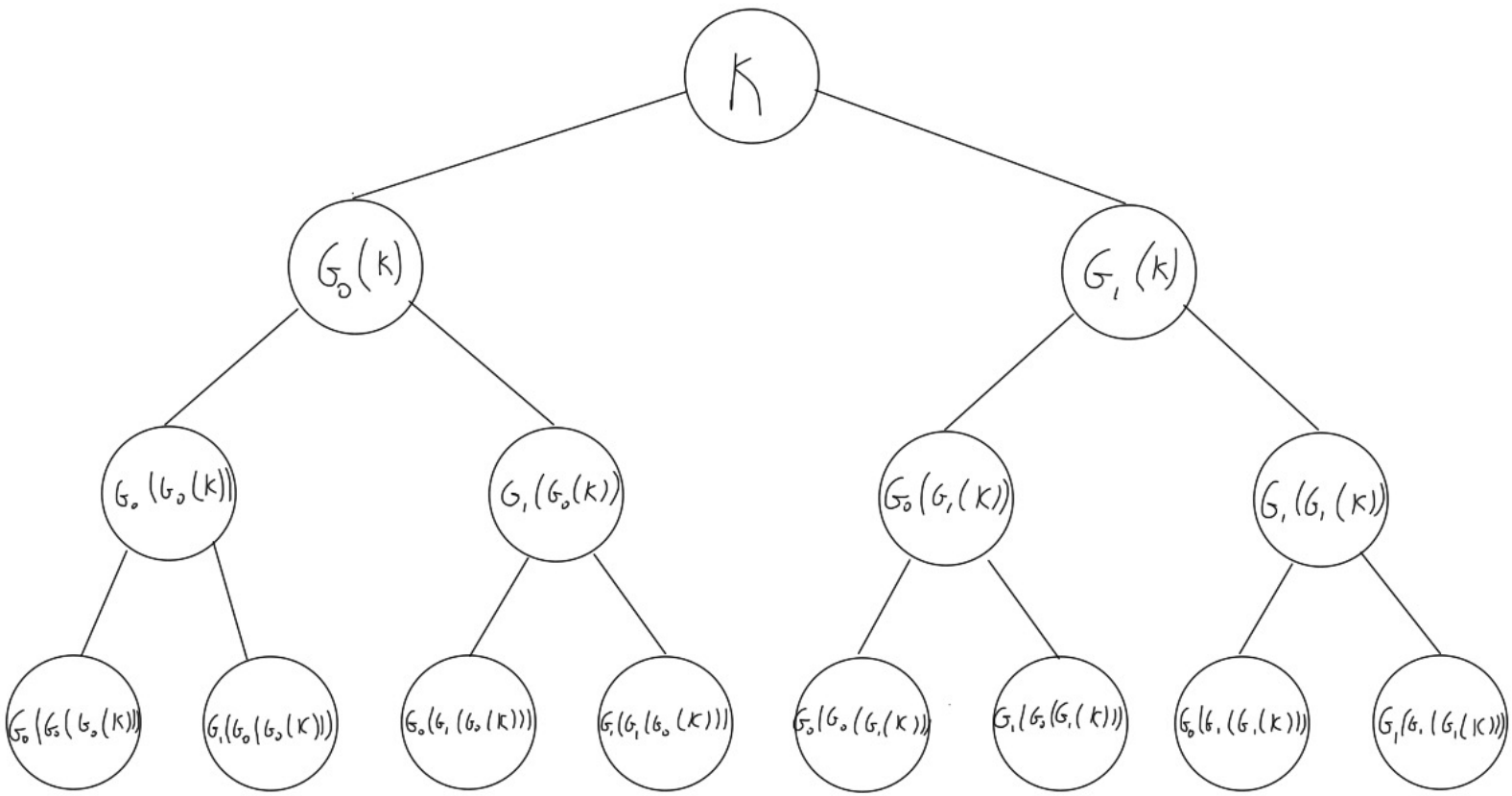

Strong PRP from PRF

Theorem

If F is a PRF, then for $k_1, k_2, k_3 \leftarrow \{0,1\}^n$, the 3-round Feistel network using $F_{k_1}, F_{k_2}, F_{k_3}$ as round functions is a strong pseudorandom permutation.

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If MACs exist (supporting an unbounded, polynomial number of queries), then one way functions exist.