CS483: Analysis of Algorithms
Flow Networks

Prof. Ivan Avramovic, GMU

Max Flow

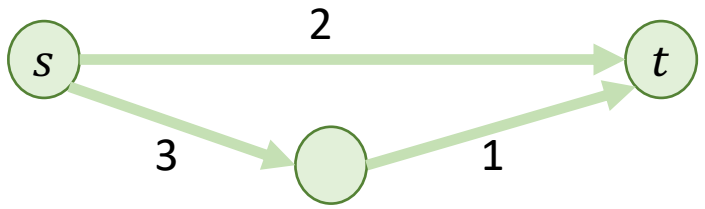
Imagine that in a weighted graph, our edge weights are **capacities**.

The capacities indicate **how much of something** can pass across an edge.

- **Example**: Amount of traffic.
- **Example**: Amount of network data.
- **Example**: Amount of liquid.

Goal of a *max flow* problem:

Find how much flow can be sent across the network without violating capacities.



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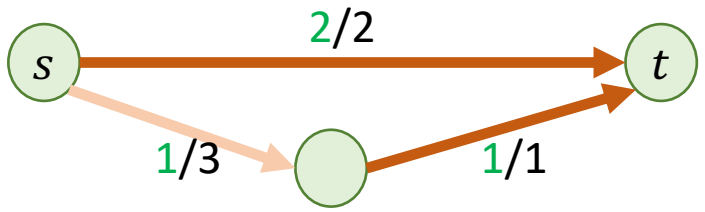


A flow of 2 along the upper path and 1 along the lower path gives a total flow of $2 + 1 = 3$.

Flow Definitions

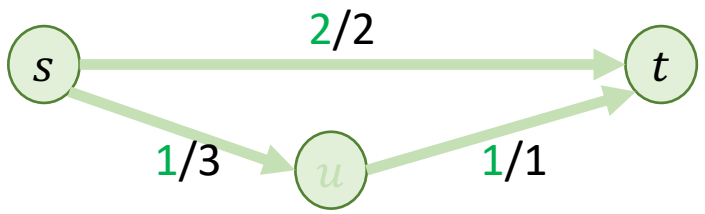
In a **flow network** for a directed graph $G = (V, E)$:

- Node $\boldsymbol{s}$ is the *source node* and $\boldsymbol{t}$ is the *sink node*.

  The flow originates from node $\boldsymbol{s}$ and flows into node $\boldsymbol{t}$.

- For any edge $e \in E$, its **capacity** is given by c_e.
- An *s-t flow* is a function f **defined over the edges** of the graph.

  Intuitively, a flow represents how much "stuff" is flowing across the network on that particular edge.



In this example, $c_{(s,u)} = 3$ and $f\big((s,u)\big) = 1$.

Max Flow, Objective

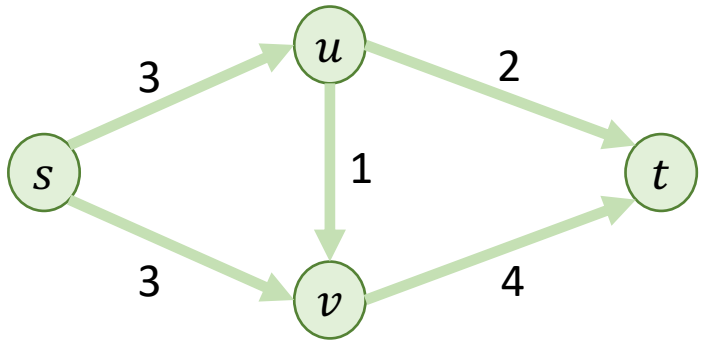
The *value* of a flow: total flow **leaving the source** or **entering the sink.**

Value of the flow $= \sum_{e \in (all\ edges\ leaving\ s)} f(e) = \sum_{e \in (all\ edges\ entering\ t)} f(e).$

Note that this will be the **same number** either way!

A max flow problem tries to **maximize** the value subject to constraints.

Assume that **all capacities are positive integers**. This is a very hard problem to solve if not.



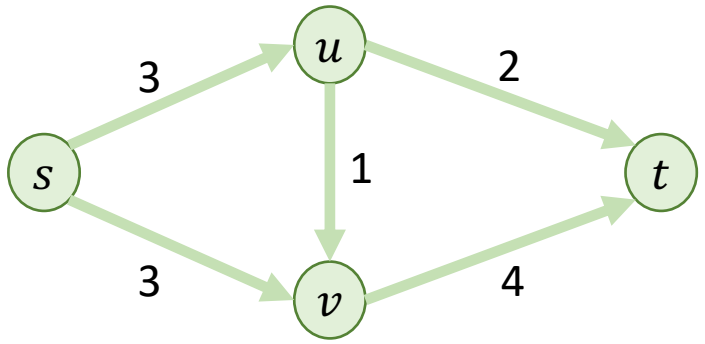
Max Flow, Constraints

Flow is **conserved** at each node other than s and t.

$$\sum_{e \in (edges\ entering\ u)} f(e) = \sum_{e \in (edges\ leaving\ u)} f(e)$$

Flow must **respect capacity**.

$$0 \leq f(e) \leq c_e.$$



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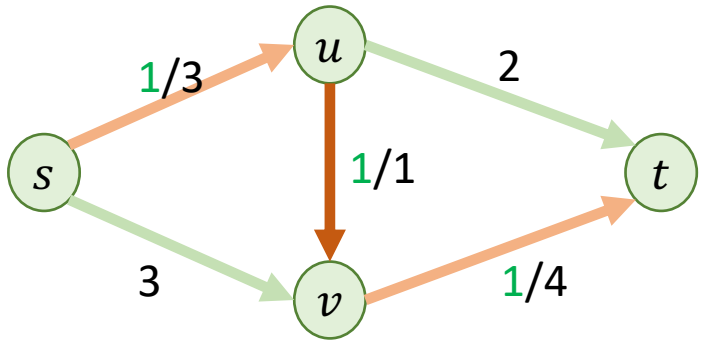

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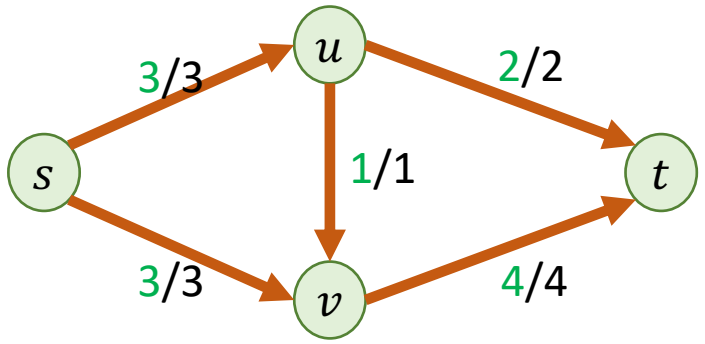
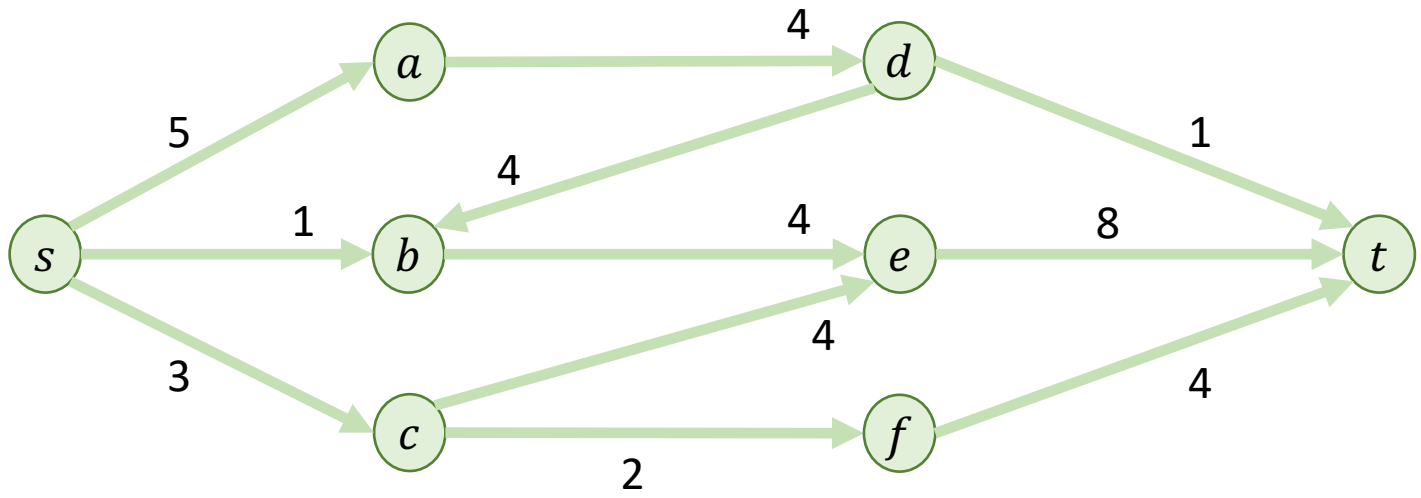


In this example, the max flow will **saturate every edge**, but this is **not always possible** in every problem.

Max Flow Example

What is the **max flow** of the following network?



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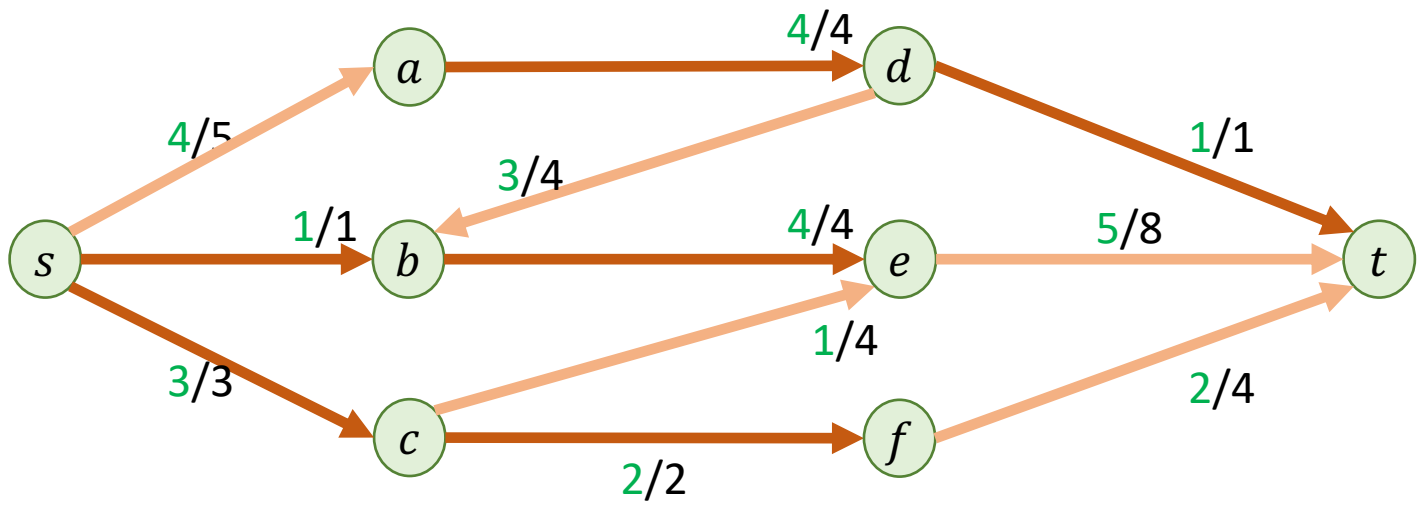
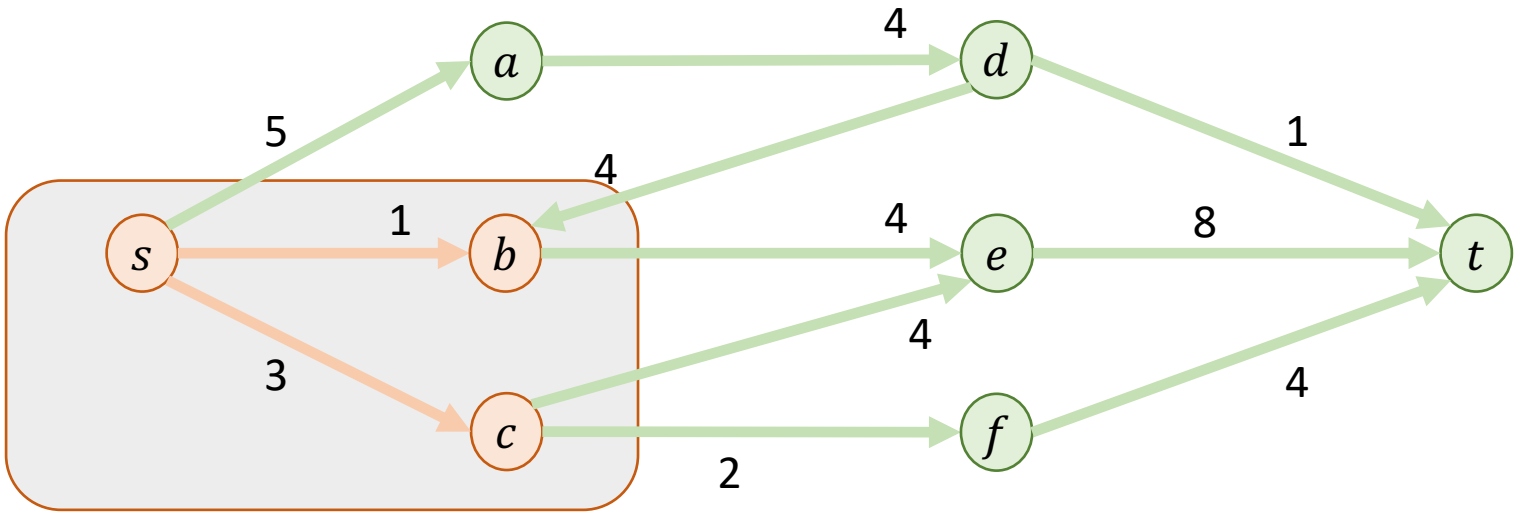


Example: Max flow value is 8.

Capacity of a Cut

A *cut* is a partition into **two sets of nodes**: s on one side; t on the other.

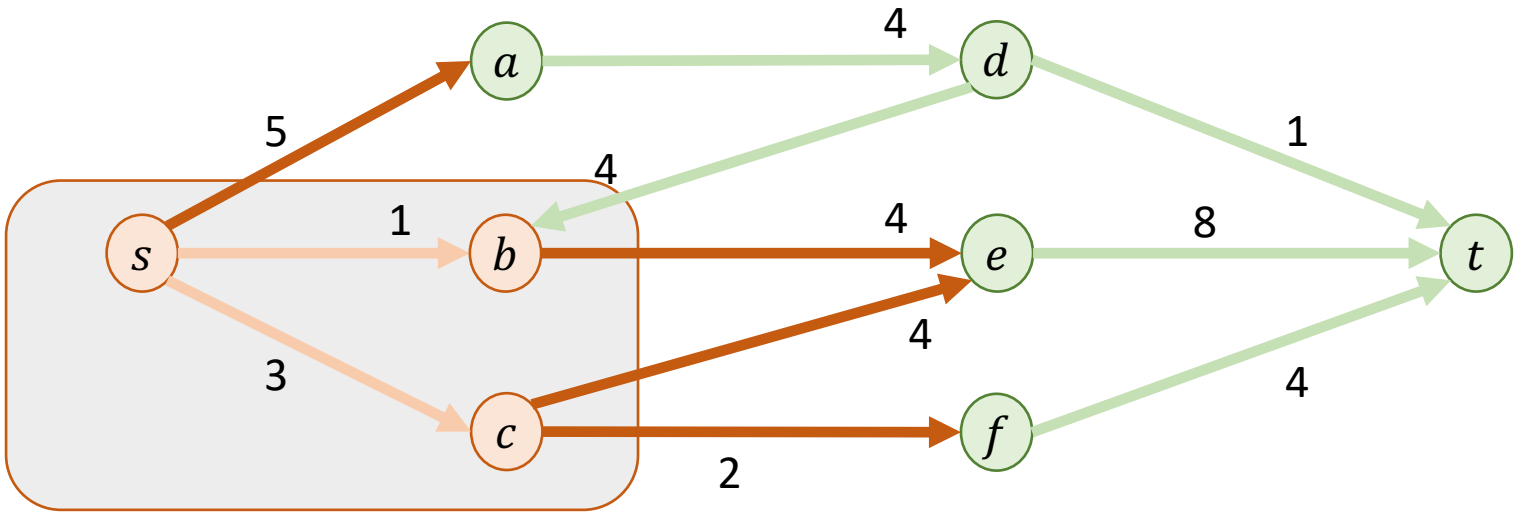
This is defined similarly to when we used cuts while computing **minimum spanning trees**.



The nodes inside a cut do not have to be touching.

Capacity of a Cut

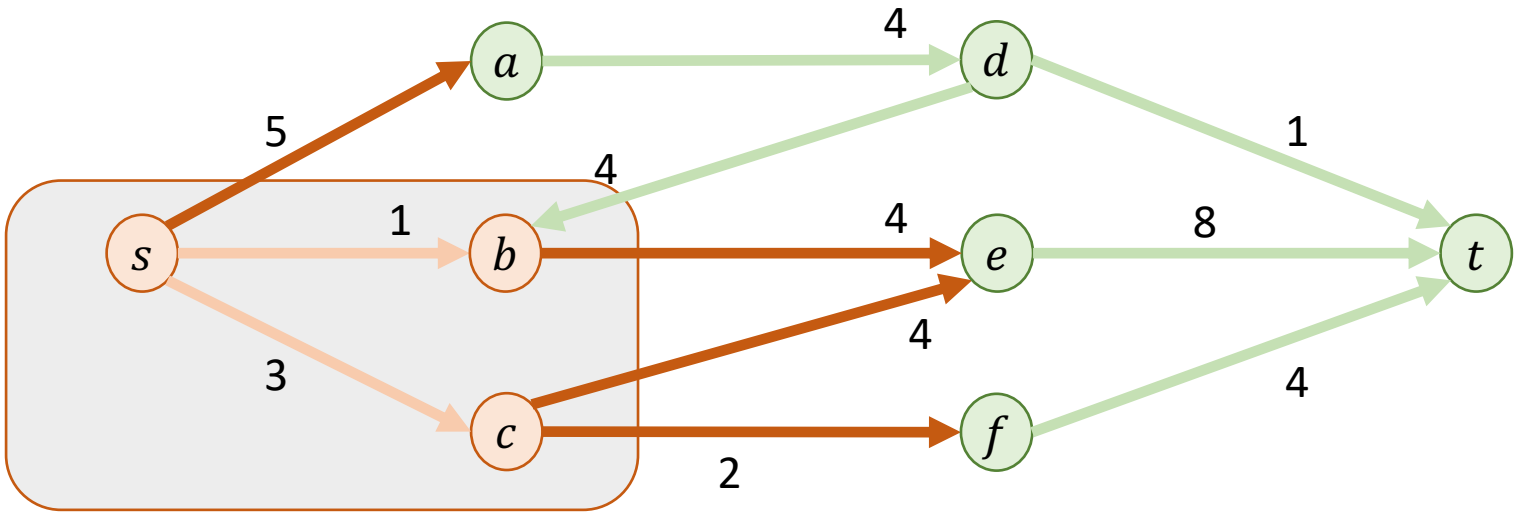
The *capacity of a cut* is the sum of the capacities **leaving a cut**.



Example: Capacity of the cut $= 5 + 4 + 4 + 2 = 15$.
Note that **edge** (d, b) **does not count** against the capacity.

Min Cut Problem

The *min cut problem* asks what is the **minimum capacity** over all cuts?



Example: What is the **bottleneck** in a data network?

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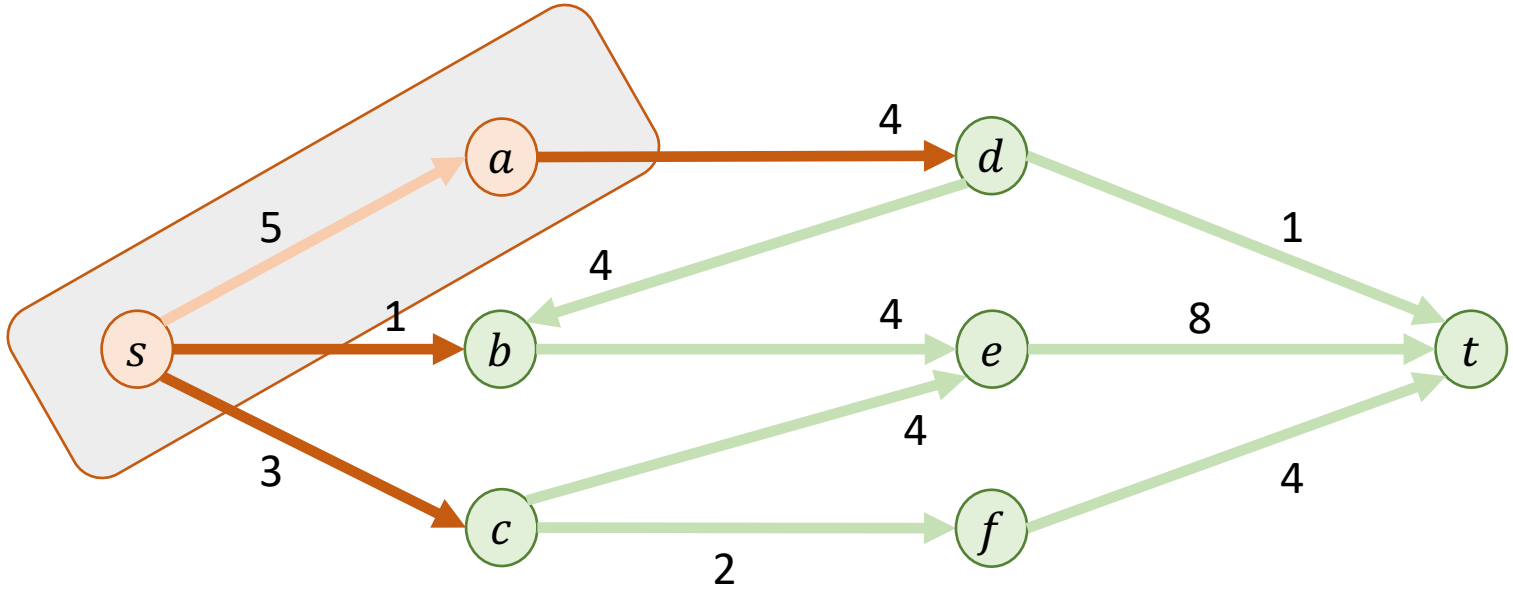
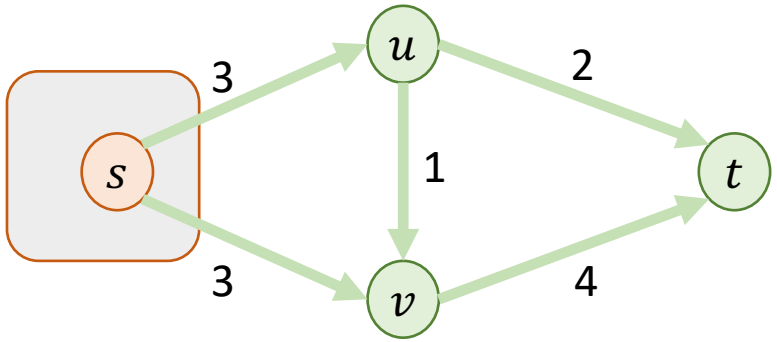


Example: Capacity $= 4 + 1 + 3 = 8$.

Min Cut Example

Example: What is the **capacity** of the cut shown?



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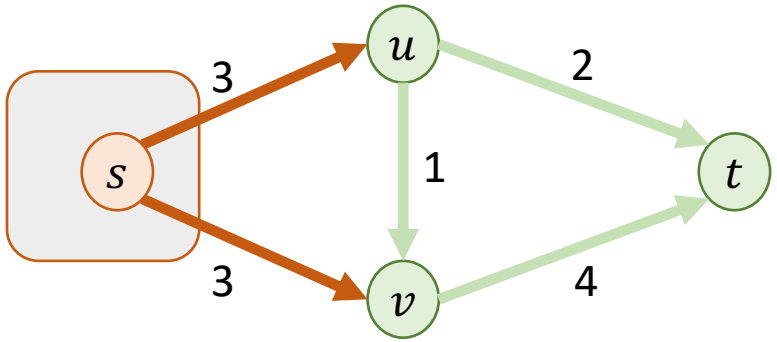

Answer: $3 + 3 = 6$.

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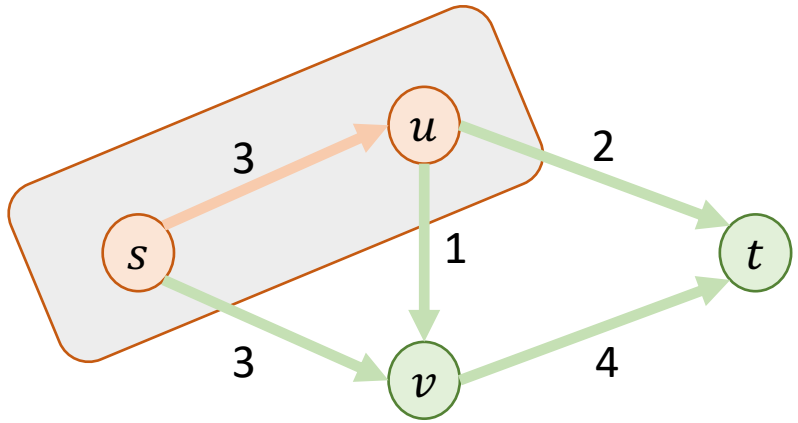

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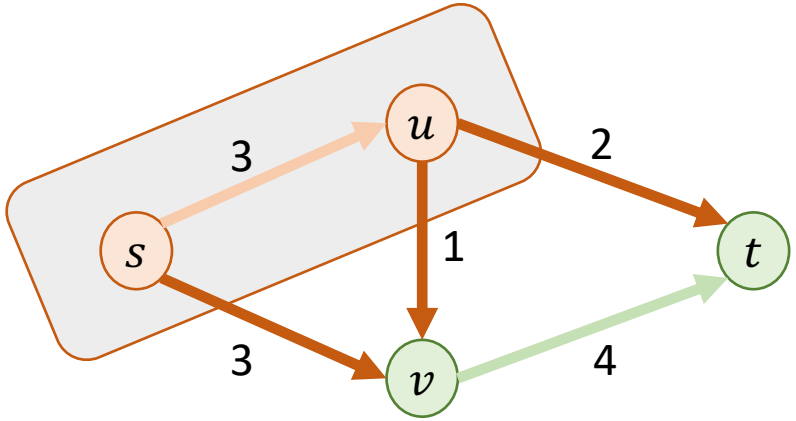

Answer: $3 + 1 + 2 = 6$.

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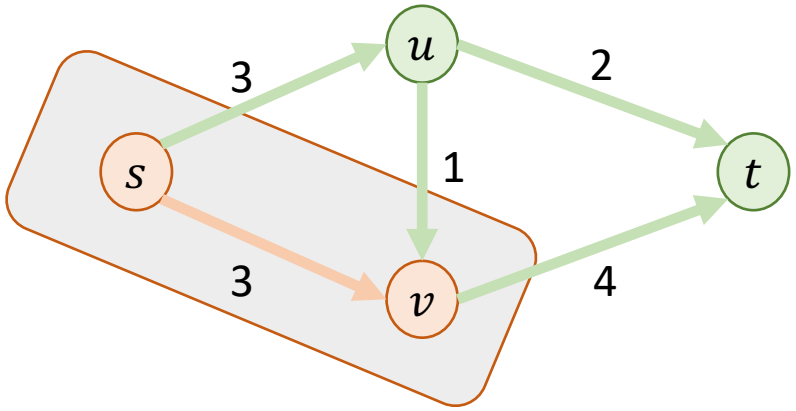

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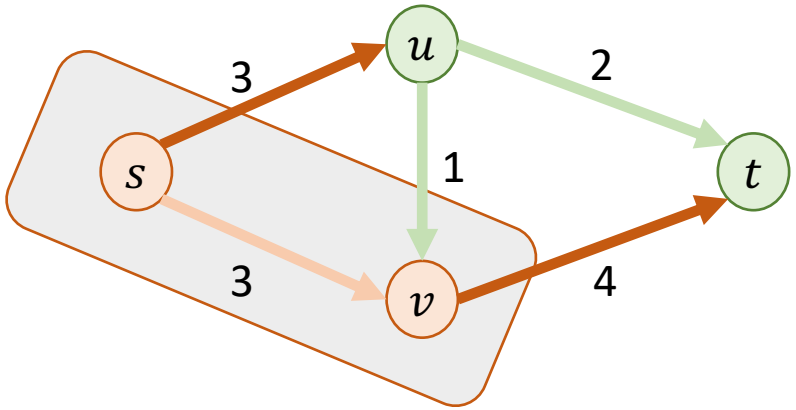

Answer: $3 + 4 = 7$.

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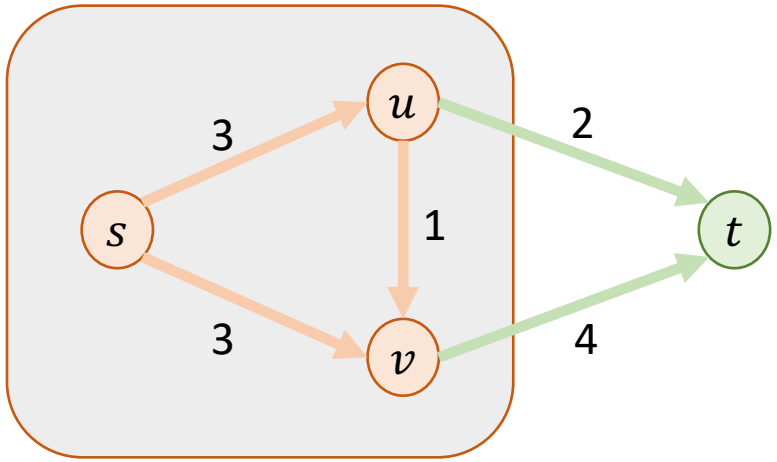

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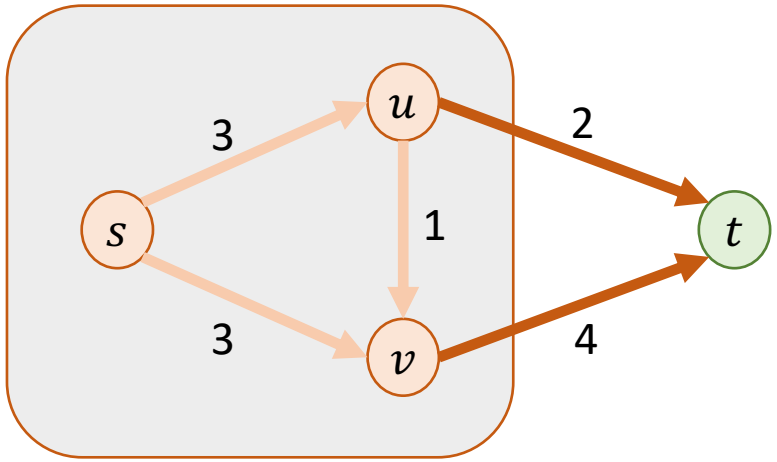
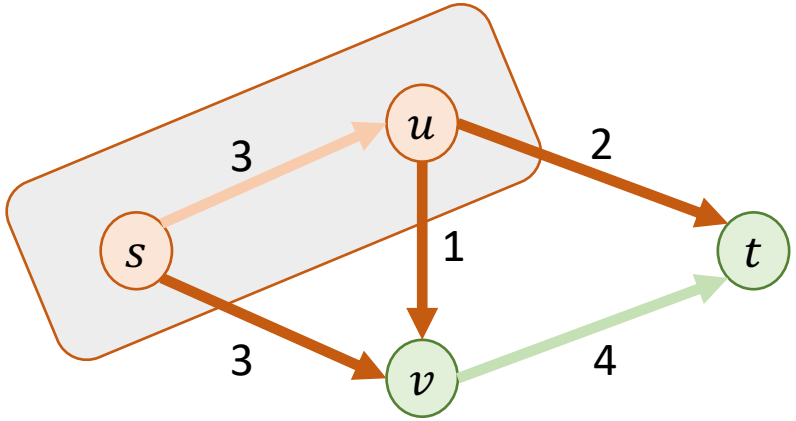


Answer: $2 + 4 = 6$.

Min Cut Example

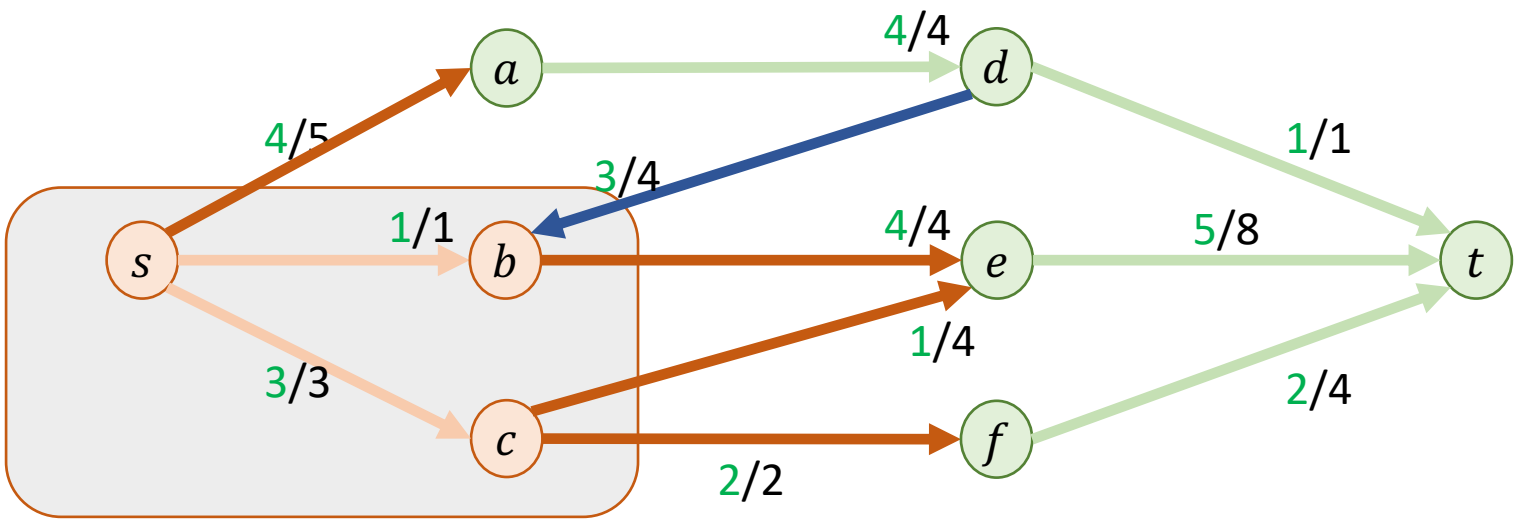
Example: The **min cut** for this network is 6, because of all possible cuts, these were the smallest.



Note that **multiple cuts** may have the **same** min cut capacity.

Net Flow Across a Cut

Net flow across a cut: difference between outgoing and incoming flows.



Net flow $= \sum_{e \in (edges\ leaving\ cut)} f(e) - \sum_{e \in (edges\ entering\ cut)} f(e)$

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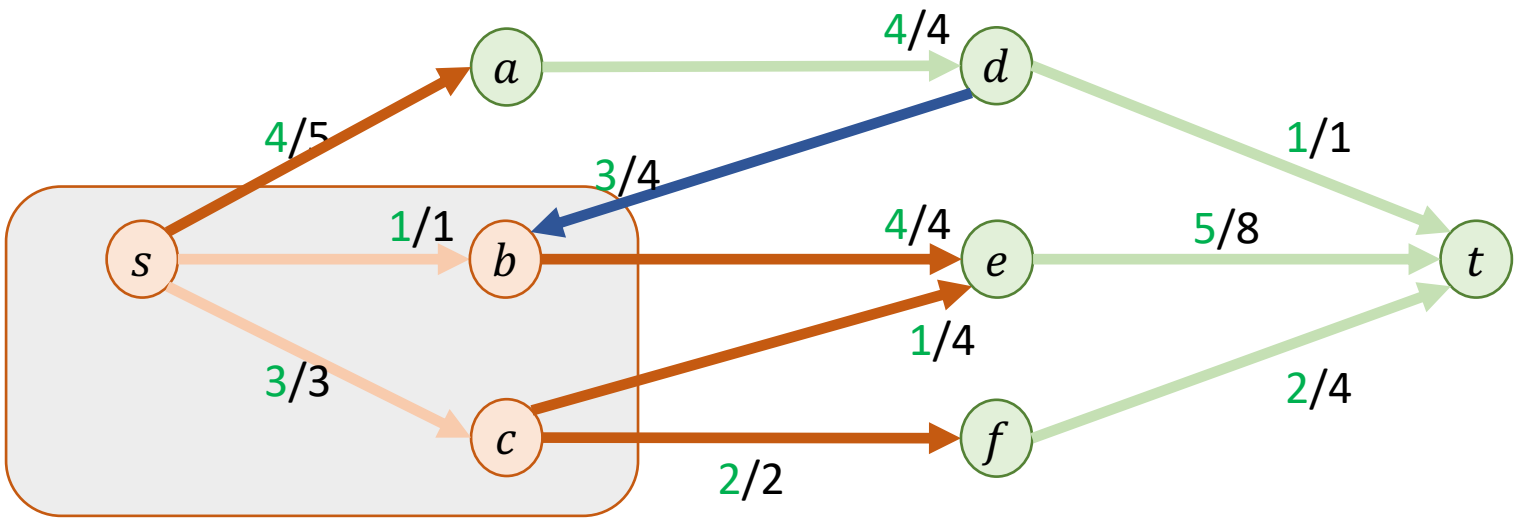
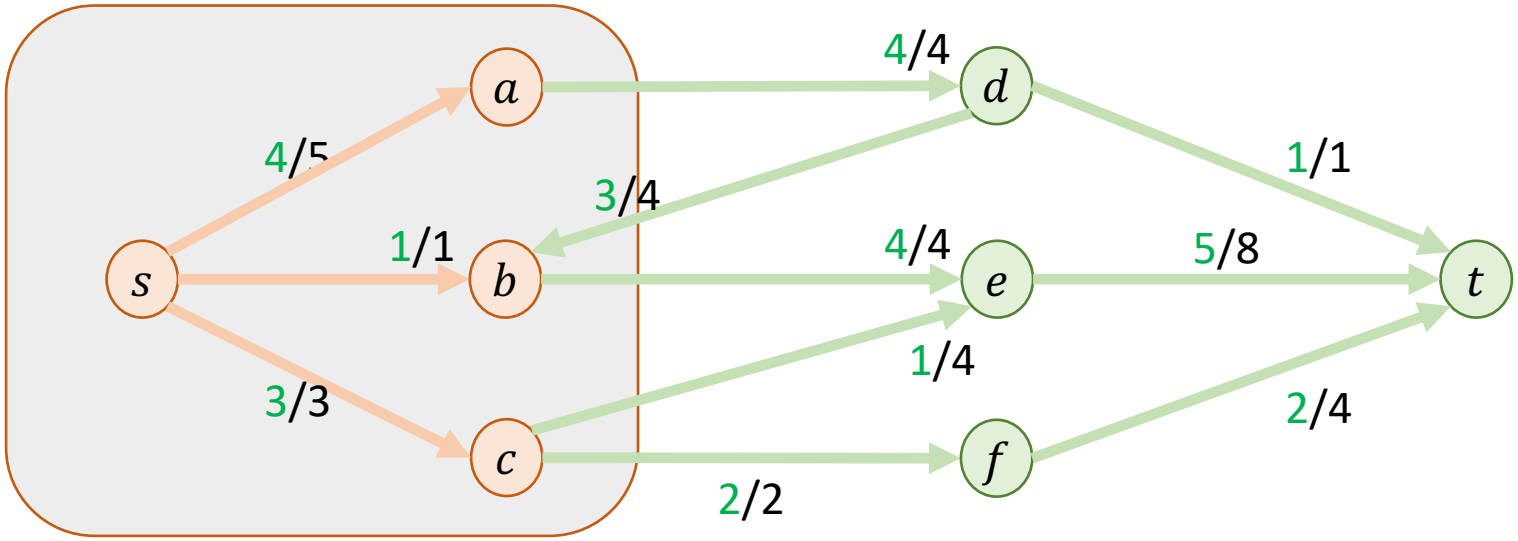


Net flow $= \sum_{e \in (edges\ leaving\ cut)} f(e) - \sum_{e \in (edges\ entering\ cut)} f(e)$

Example: Net flow $= (4 + 4 + 1 + 2) - (3) = 8.$
Capacity $= 5 + 4 + 4 + 2 = 15.$

Net Flow Example

What is the **net flow** across the following cut?



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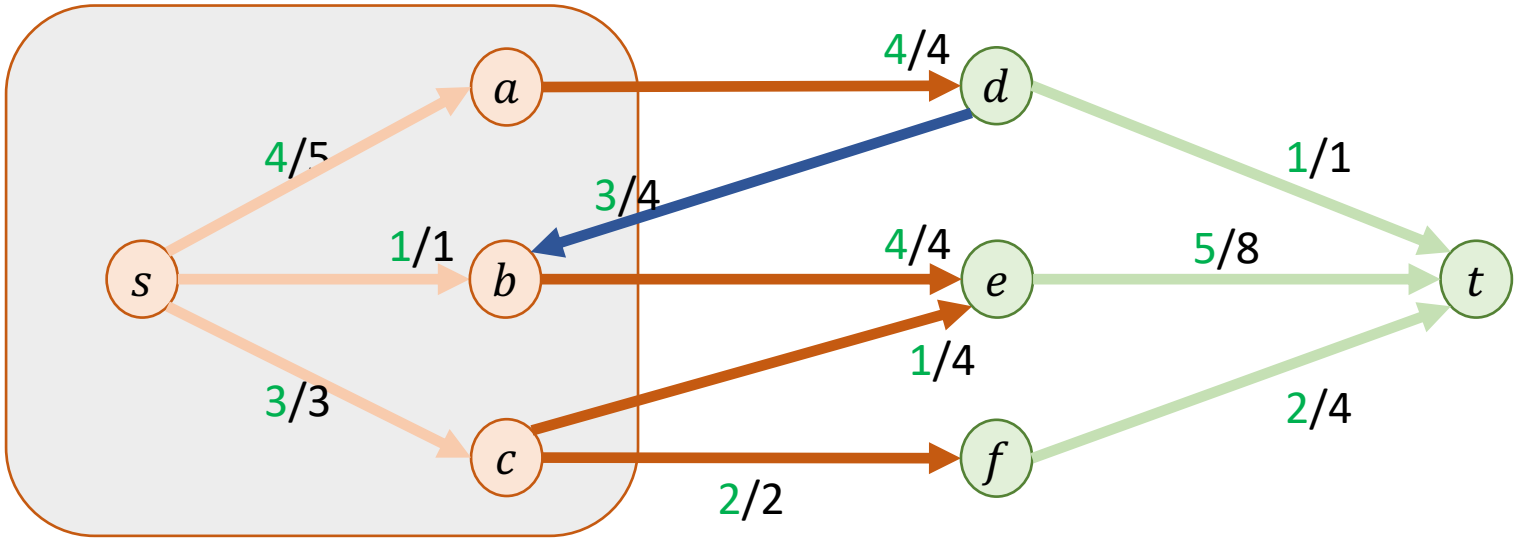

Answer: Net Flow $= (4 + 4 + 1 + 2) - (3) = 8$.
Capacity $= 4 + 4 + 4 + 2 = 14$.

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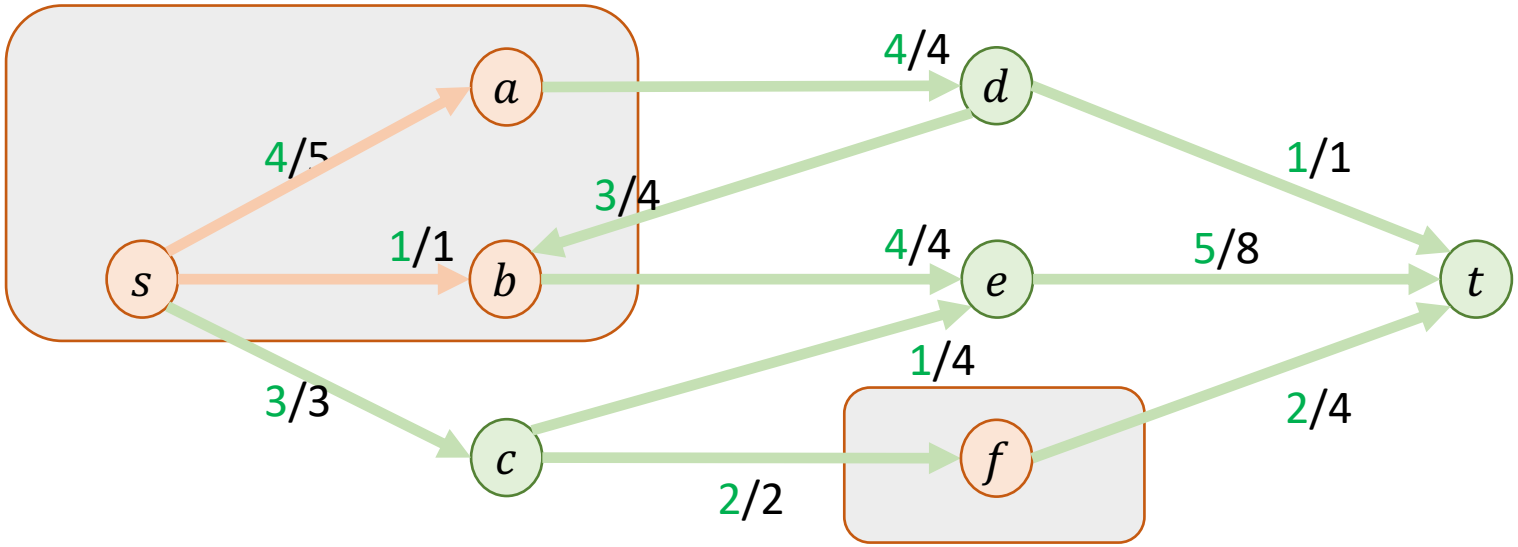

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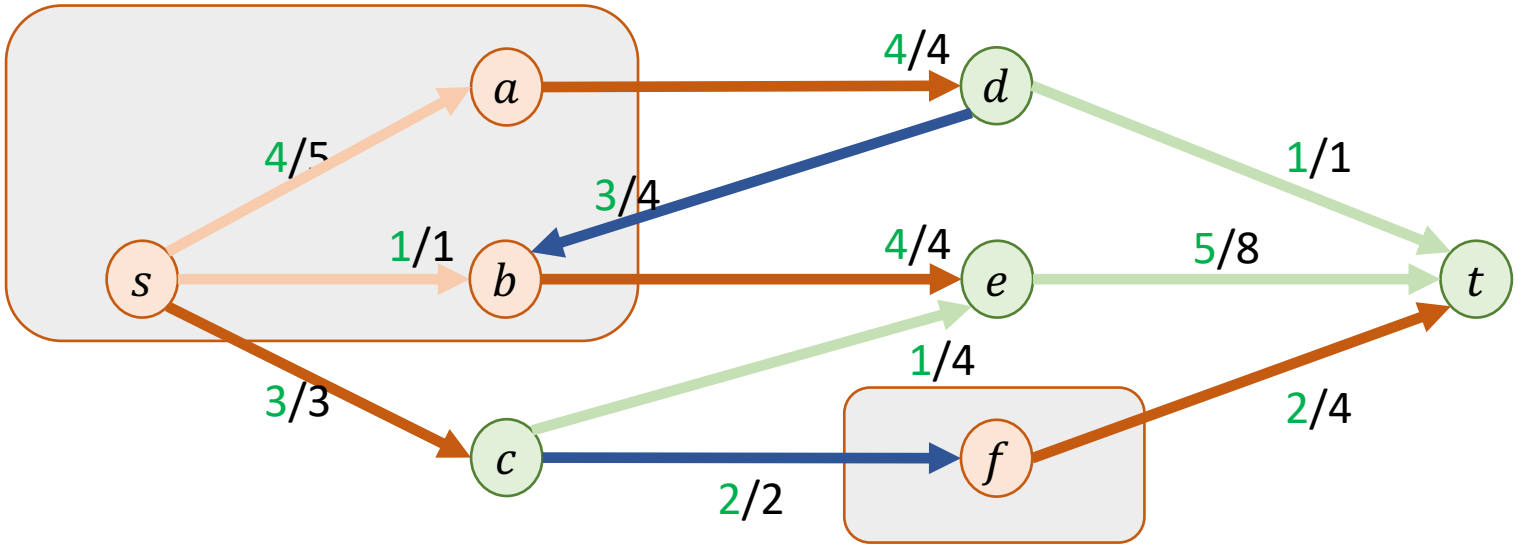
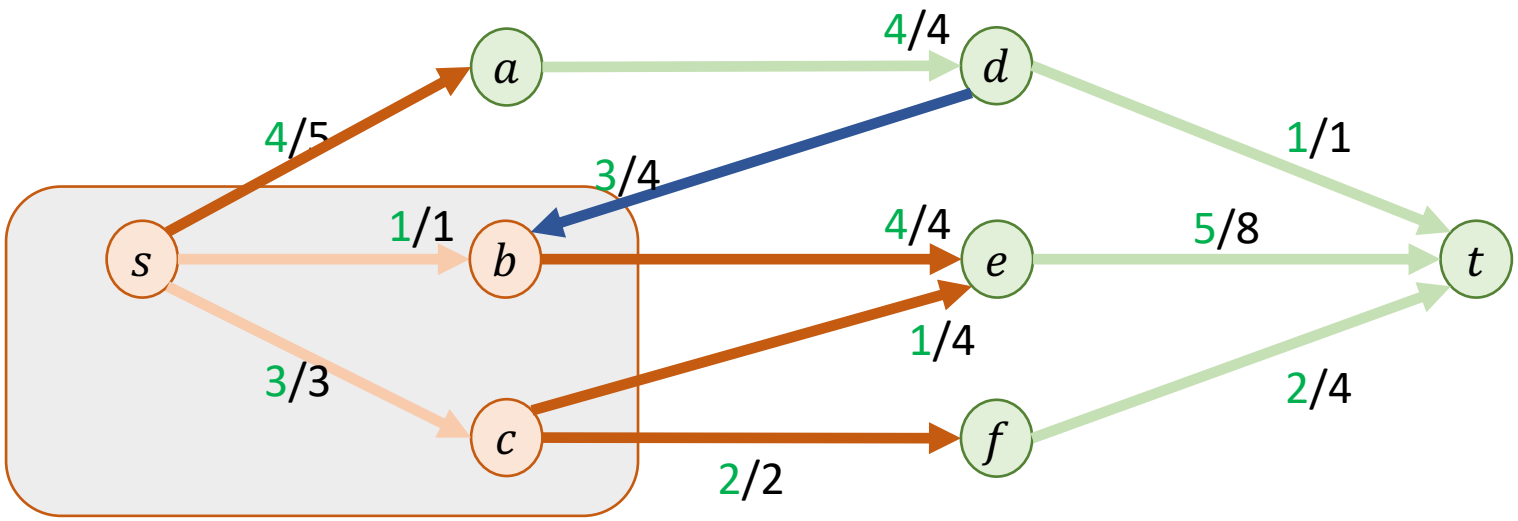


Answer: Net flow $= (4 + 4 + 3 + 2) - (3 + 2) = 8.$
Capacity $= 4 + 4 + 3 + 4 = 15.$

Net Flow Across a Cut

Net flow across a cut: difference between outgoing and incoming flows.

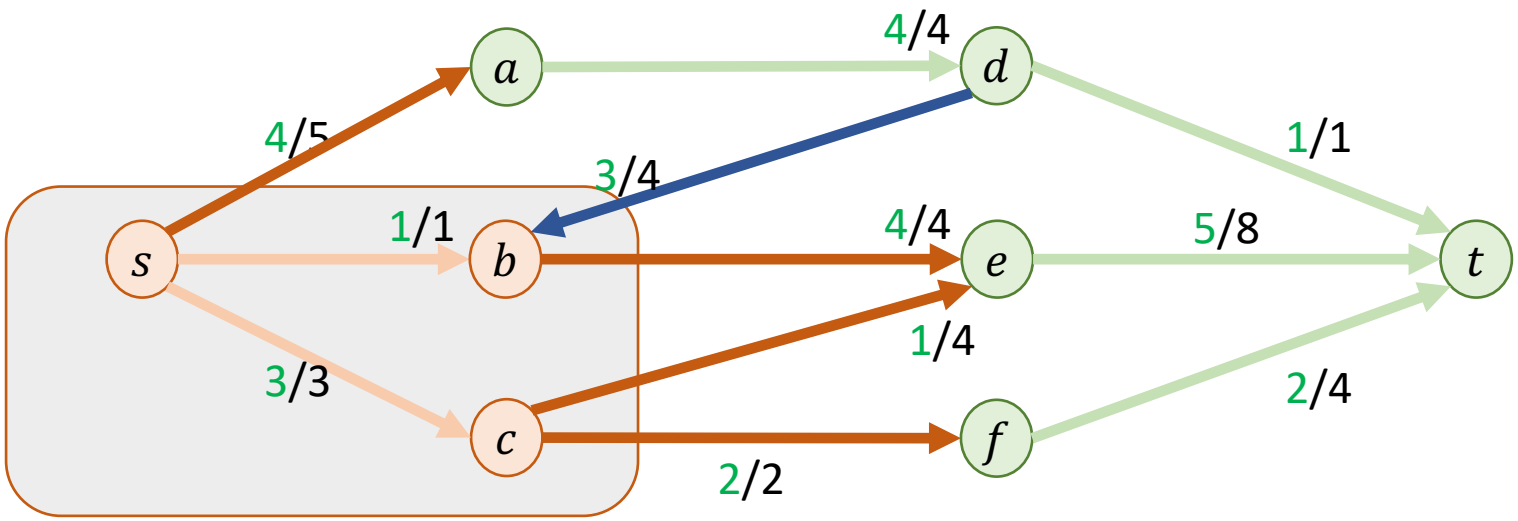


Observation 1: The **net flow will always be the same** no matter which cut we choose.

- This can be deduced from the **conservation** principle as we add or remove nodes.

Net Flow Across a Cut

Net flow across a cut: difference between outgoing and incoming flows.



Observation 2: The **max flow cannot be larger** than the capacity of a cut.

- The capacity is an **upper bound** for net flow.

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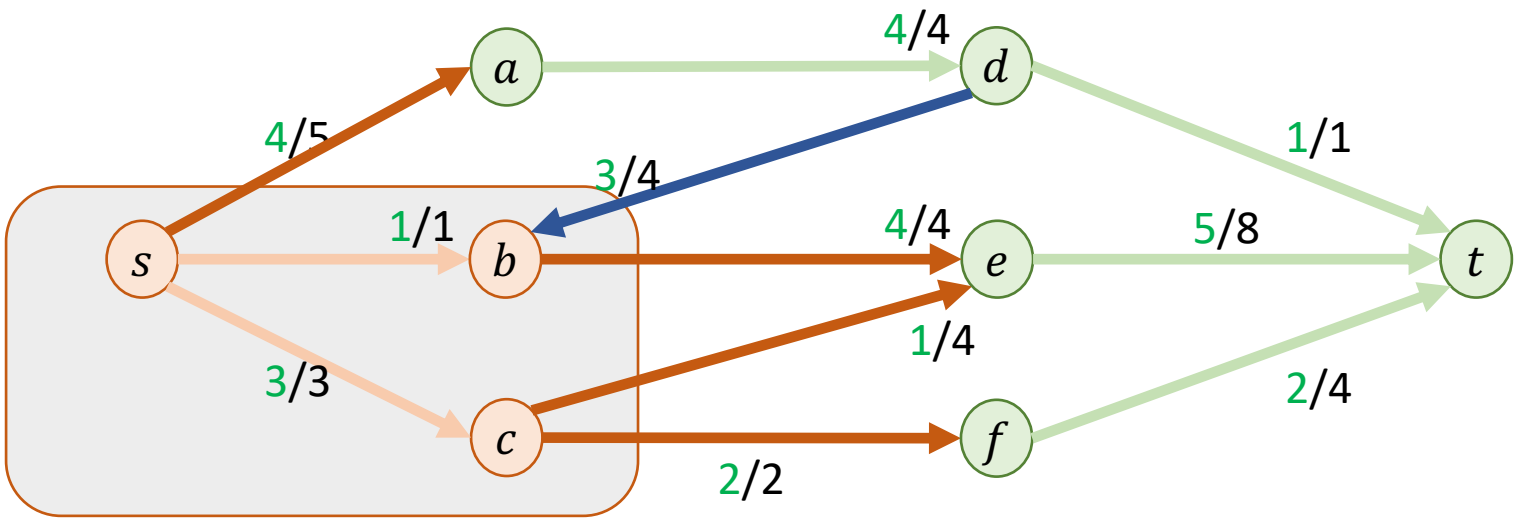


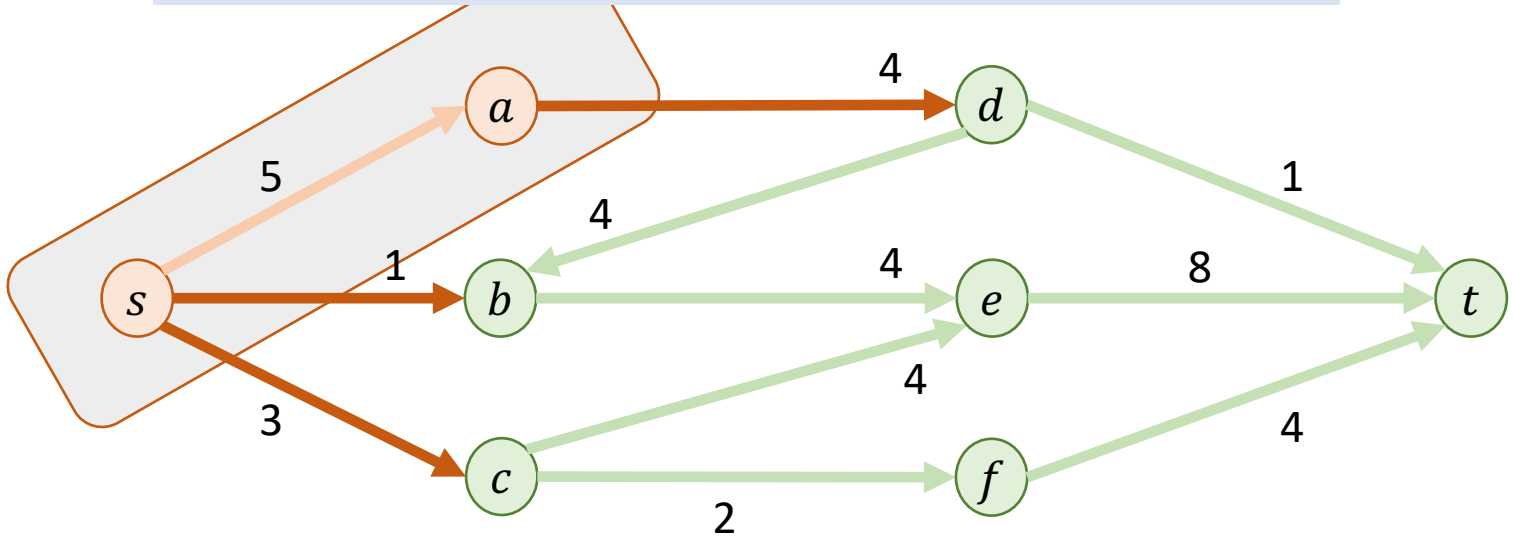
Observation 3: Some cut must have a capacity which **equals the max flow**.

- If not, then we could have still **sent more flow** across the network.

Max Flow/Min Cut Duality

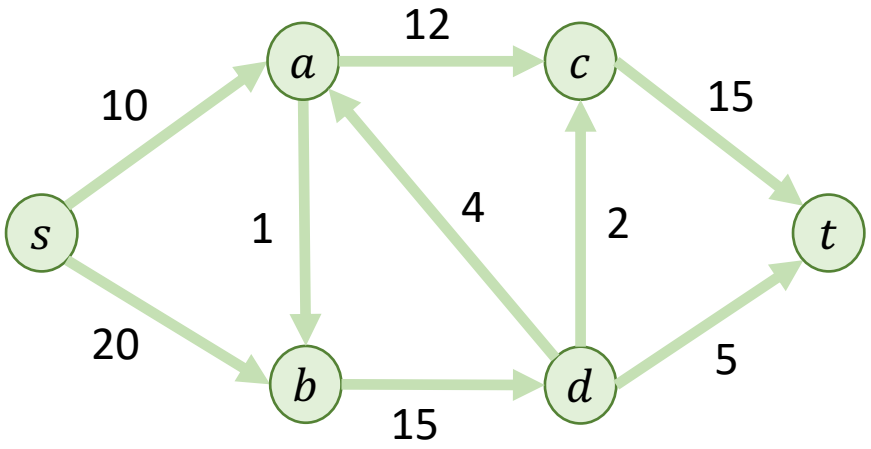
Conclusion: the **max flow** value equals the **min cut** value!

If we can compute one, then we have computed the other.



Practice

Find a **max flow** and a **min cut** for the following graph:



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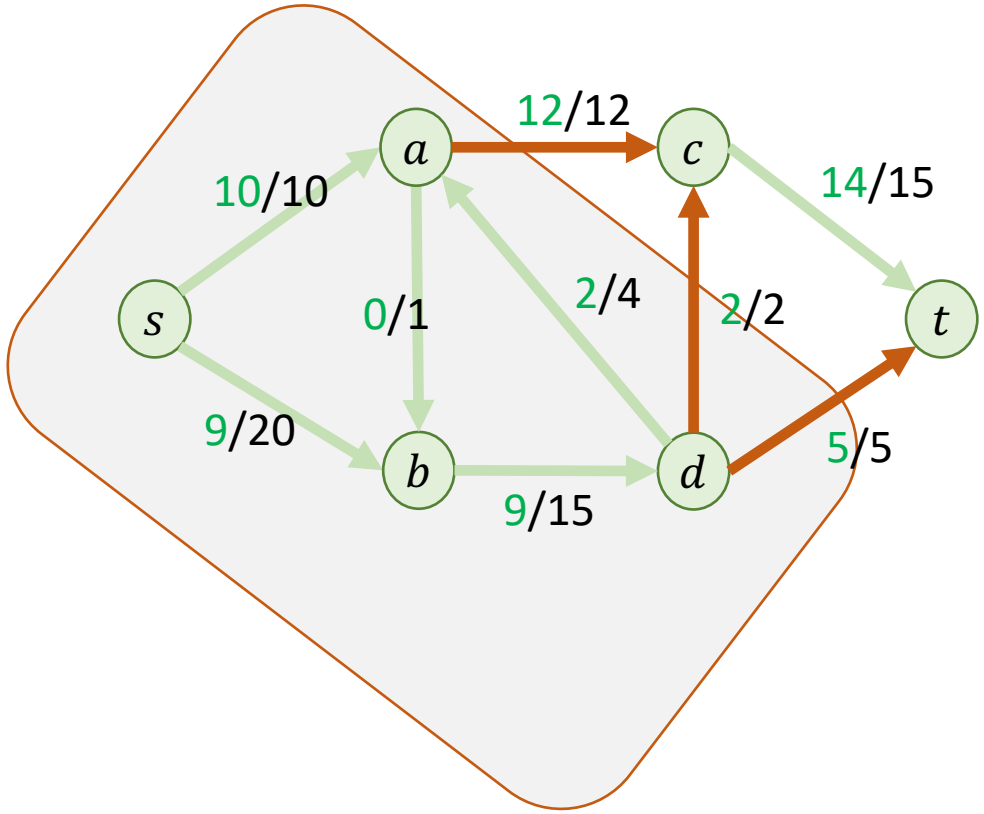


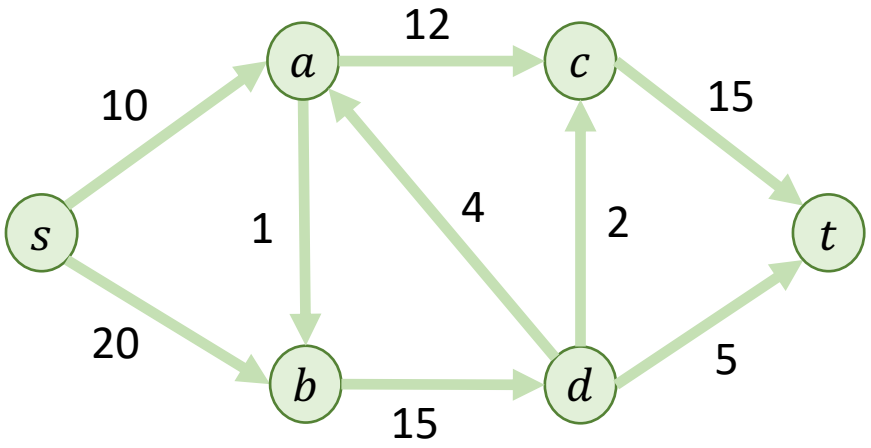
Flow value = 19

Min cut capacity = 19

Algorithmic Max Flow: Baby Steps

If we want to find max flow, we would prefer to use an **algorithm**.

Before we produce an algorithm for **max flow**, let's start simple… can we find **any** flow through this graph?



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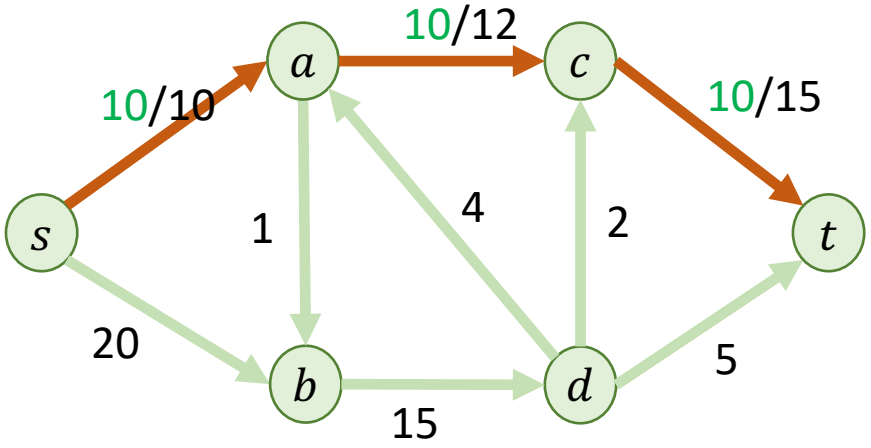
Suggestion: Let us find **any path** from source to sink. That path implies a **flow**.

How would we **find a path** from source to sink?



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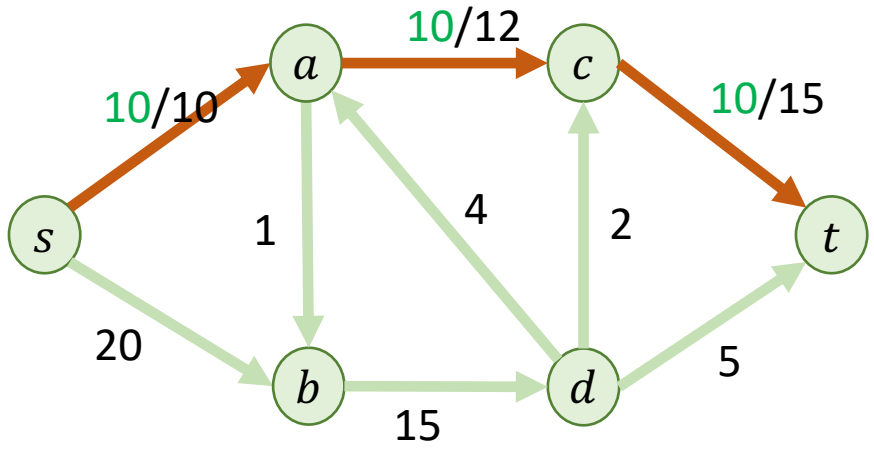

How would we **find a path** from source to sink?

Our ordinary **graph search algorithms** will find one.

- **Breadth first search** (BFS).
- **Depth first search** (DFS).

On the other hand, **if there is no path**, then there **is no possible flow**.

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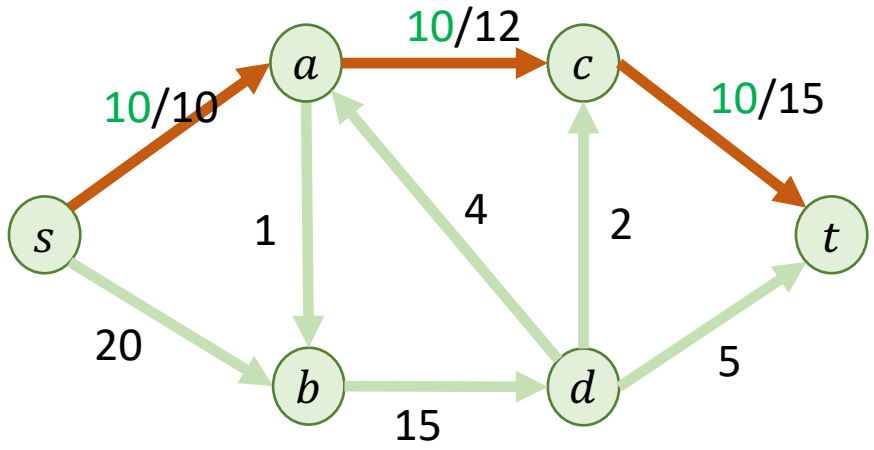


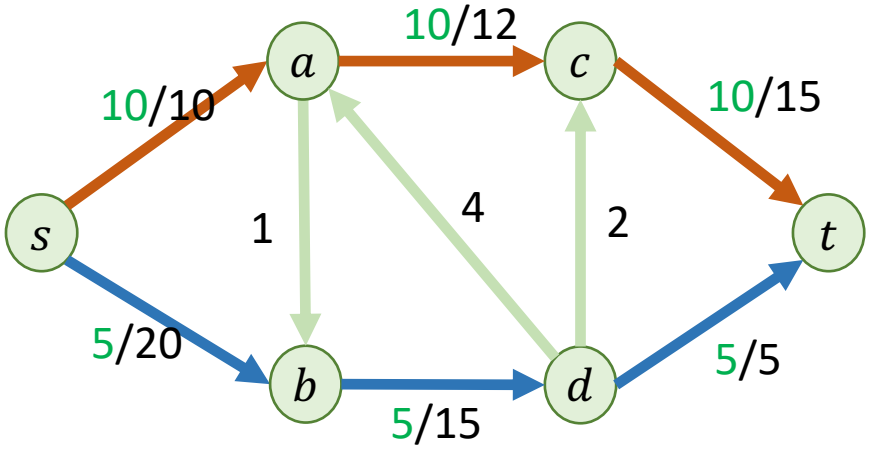
We will **prefer a BFS**, for reasons we shall see later.

If we find a path, we can use a **bottleneck edge** to **maximize flow along that path**.

Algorithmic Max Flow: Augmenting

If we want to find max flow, we would prefer to use an **algorithm**.

Observation: If we find a **second path**, we can combine it with the first to form a **larger flow**, provided that the two together do not violate any capacities.



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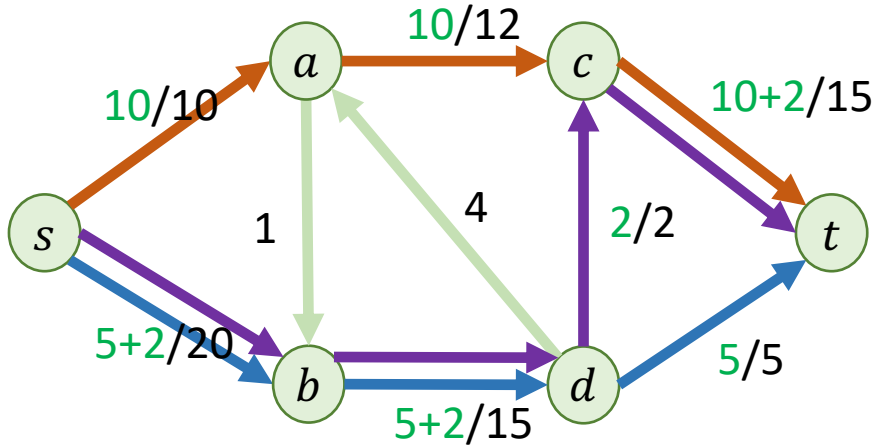
This works even if the new path **overlaps an existing path**.

These added paths are called *augmenting paths*.

- Augmenting paths will **only increase** flow value.



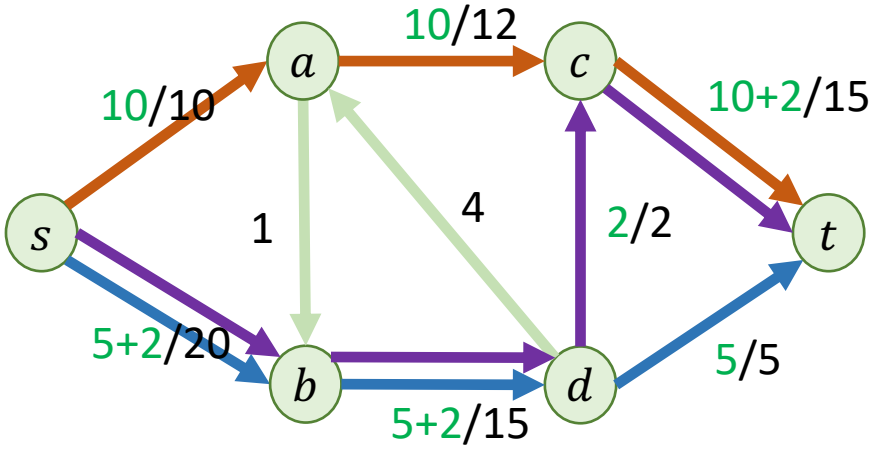
Algorithmic Max Flow: Augmenting

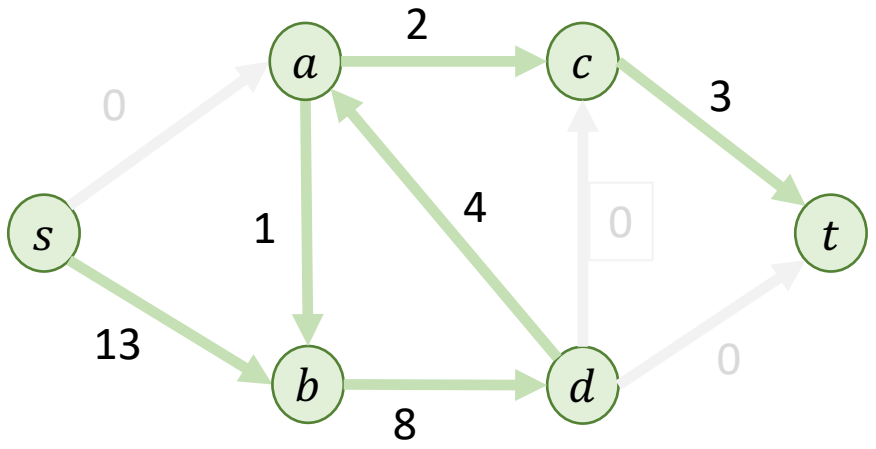
If we want to find max flow, we would prefer to use an **algorithm**.

How would we **find new paths** once we have already used some of the paths?

- We do not want to **waste time revisiting paths** which are already used.

Suggestion: Consider only the capacity which is "**left over**" after using known paths.





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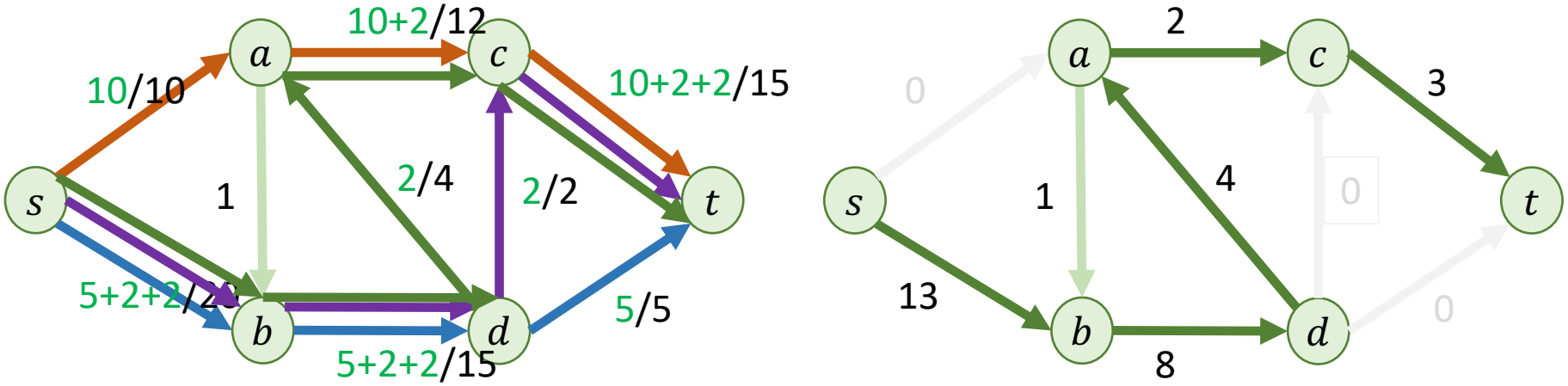
- We do not want to **waste time revisiting paths** which are already used.

Suggestion: Consider only the capacity which is "**left over**" after using known paths.



Algorithmic Max Flow: Augmenting

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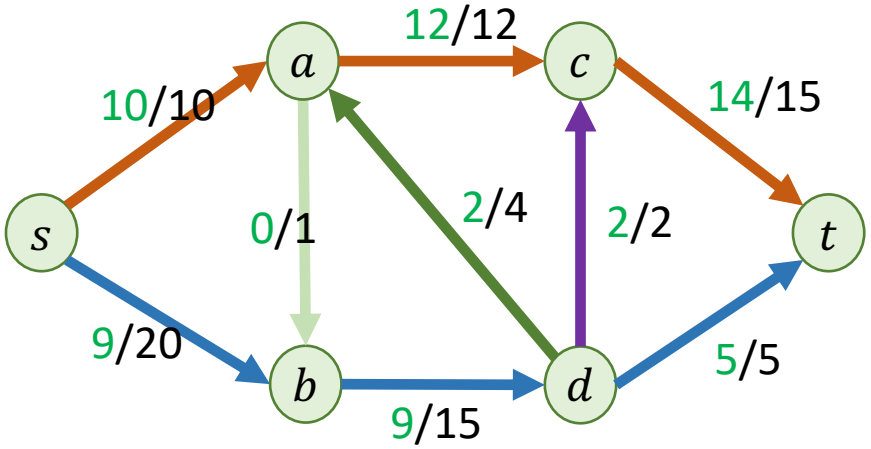
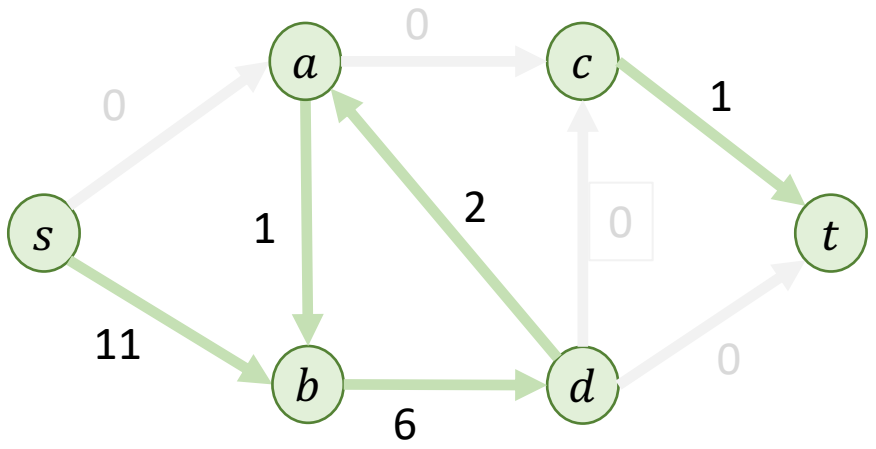


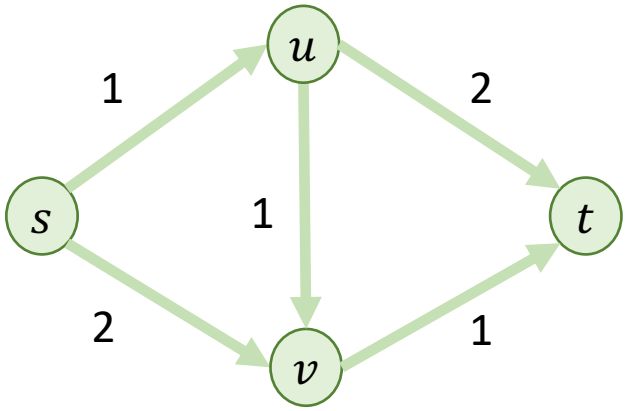


Note that there is now **no more path** from source to sink.

Reliably Finding Augmenting Paths

If we always **removed used capacity** from the graph whenever we find a flow, will that always **lead us to a max flow**?

Maybe that is not enough. **Consider**:



What is the max flow here?

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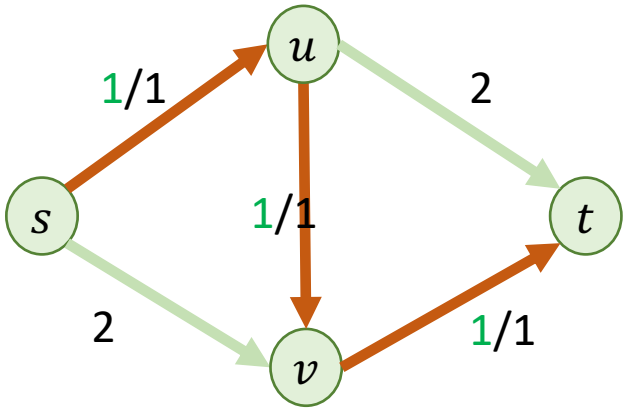

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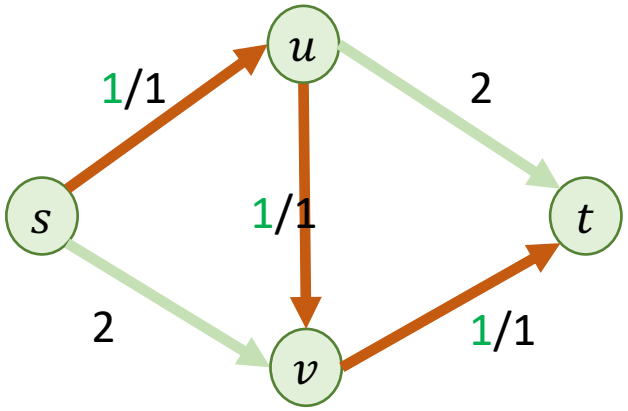

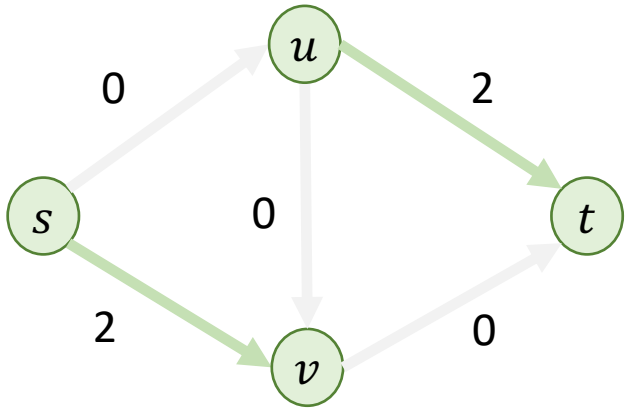

What is the max flow here?

There are **no more paths to find**, yet we can easily see that the true **max flow is better than this**!

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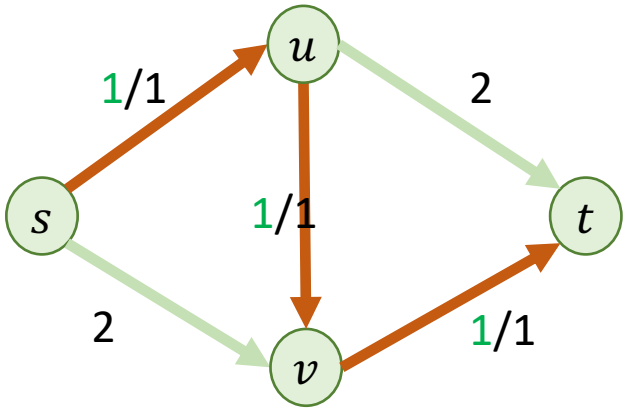

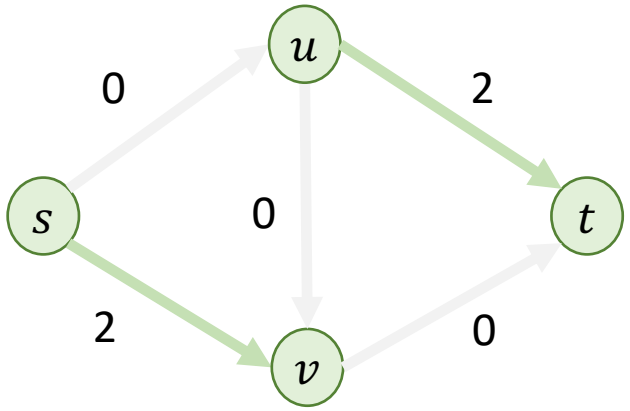

What is the max flow here?

The best result does not use the middle (u, v) edge.
Is there a way for us to **undo** a selection?

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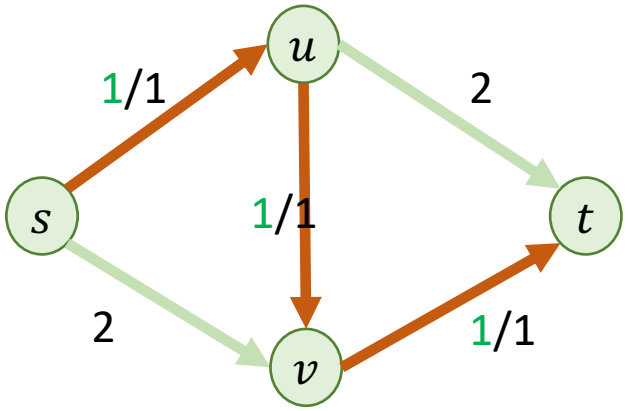

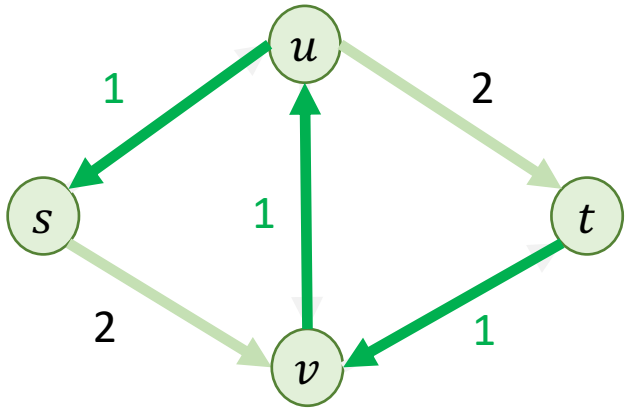

What is the max flow here?

Solution: We can create a virtual edge in the **opposite direction** to show **how much less flow** we could have had across that edge. This lets us **undo** a selection.

It is **still true** that **any path** from source to sink **increases the flow** value.

Reliably Finding Augmenting Paths

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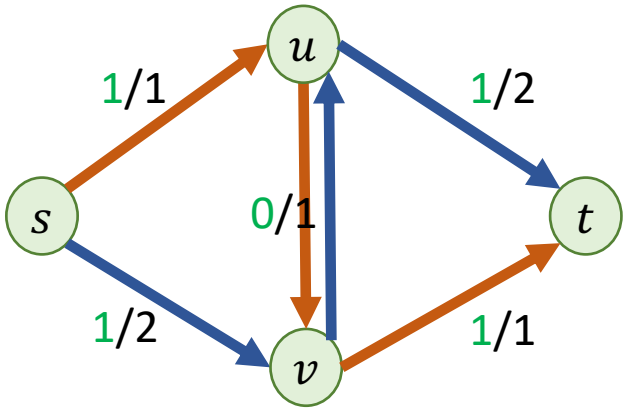

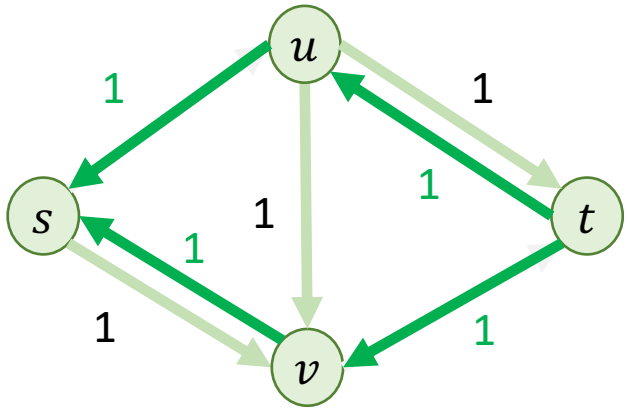

Now, our "left over" graph – called a *residual graph* – has **no remaining *s*-*t* path**.

- We can truly say that **no more flow** can be added – we have a max flow!

Ford-Fulkerson Method

The *Ford-Fulkerson Method* for **finding a max flow** is as follows:

Repeat:

- Find an **augmenting path** (e.g. using a search algorithm from s to t).
- **If found**:
  - **Add the augmenting path** to the flow.
  - Recompute a **residual graph**.

…until no more augmenting paths can be found.

Min Cut Algorithm

Now that we know max flow, can we use it to find **min cut**?

Hint: What is the **relationship between** max flow and min cut?

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The capacity at the min cut **equals the max flow** value.

- In fact, the **net flow** must be the same as well.

Hint: What is true of the **residual graph** relative to the min cut?

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- In fact, the **net flow** must be the same as well.

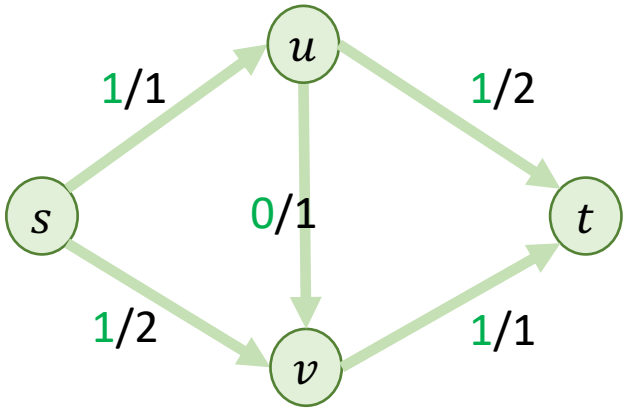
Hint: What is true of the **residual graph** relative to the min cut?

The flow must have **saturated the cut**, so the residual graph will have **no edges leaving** the cut.

- **Conclusion**: We can find a min cut by **finding max flow** and **computing the residual graph**.
- **The set of nodes which are still reachable will form a min cut.**

Min Cut Algorithm

1. Find max flow.
2.
3.



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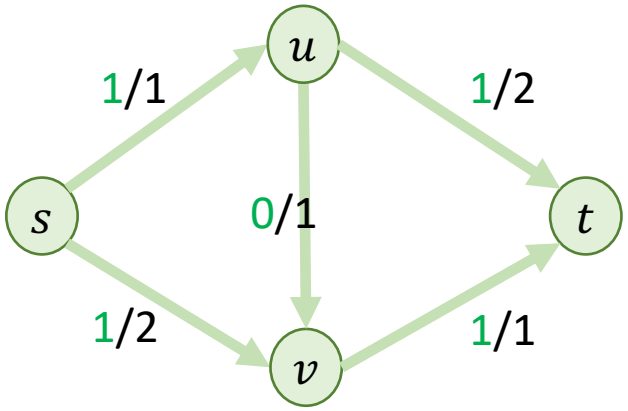
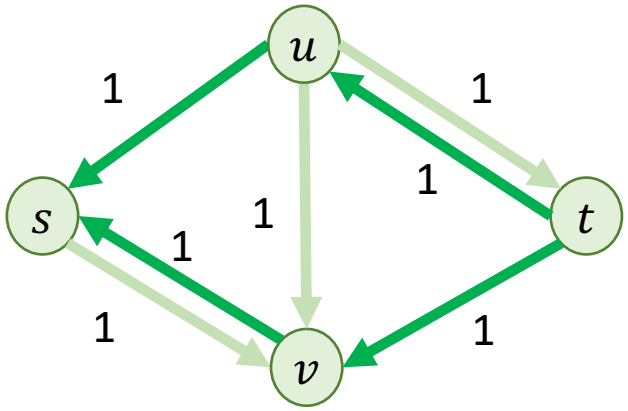
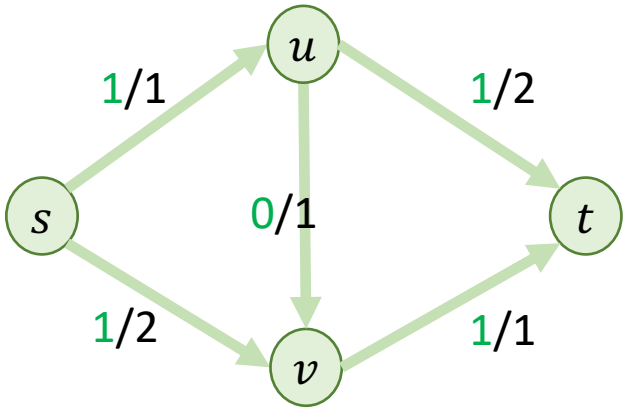
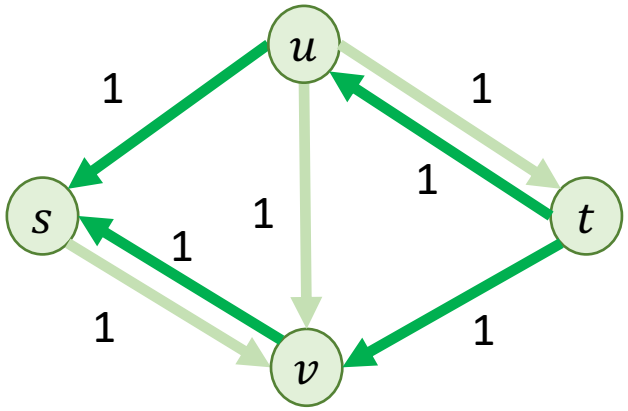


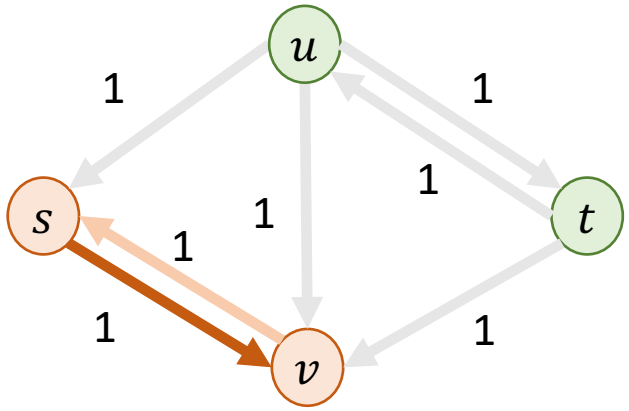


Min Cut Algorithm

1. Find max flow.
2. Compute residual graph.
3. Perform a BFS or DFS to find a min cut.

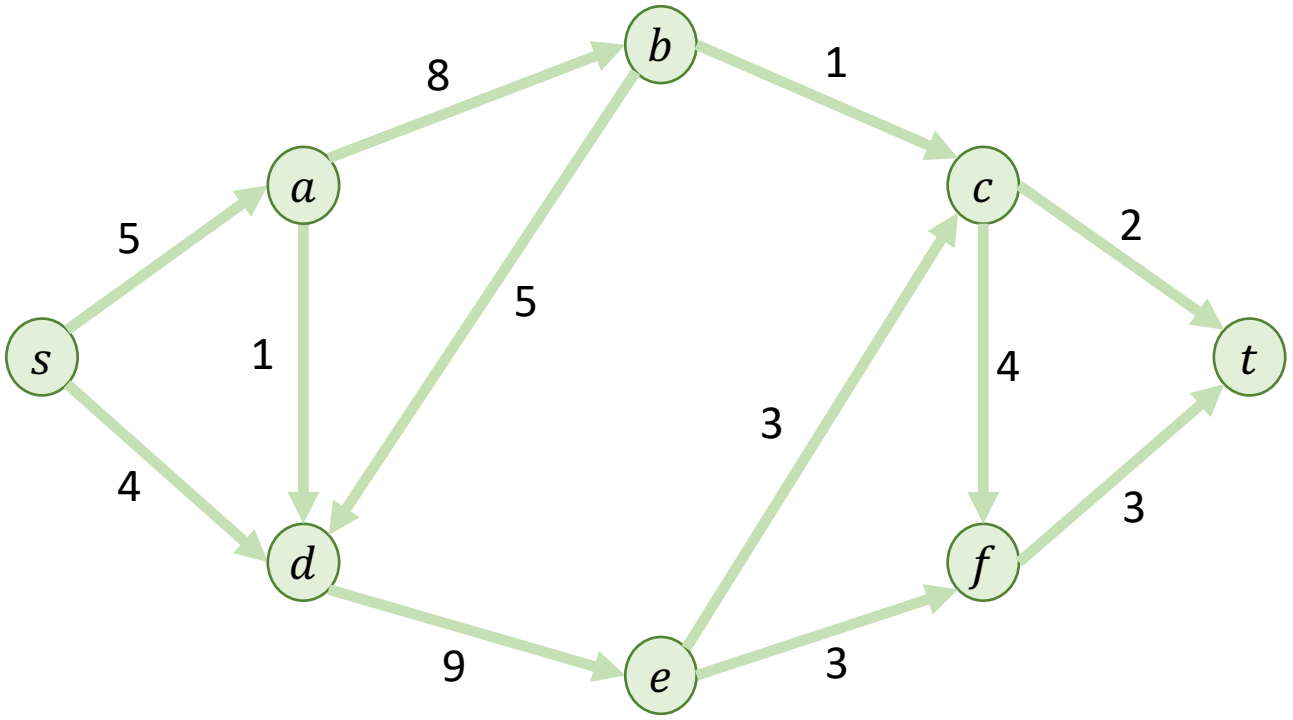






Practice

Find a **max flow**, **residual graph**, and **min cut** for the following graph:



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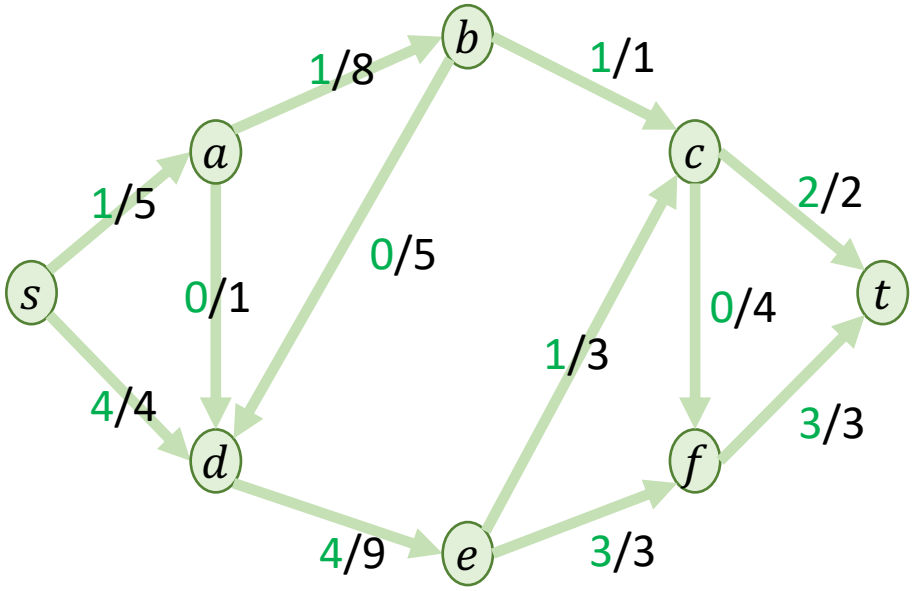
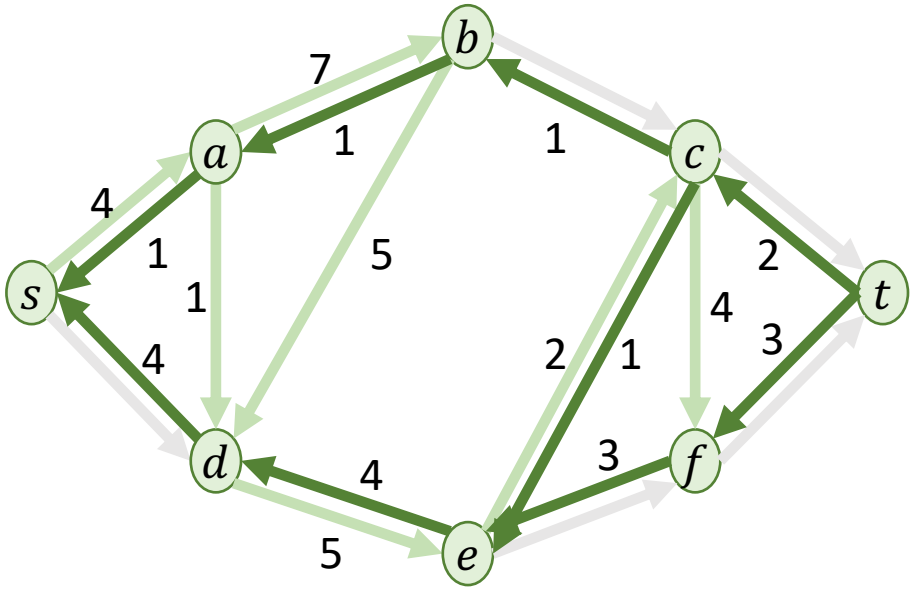


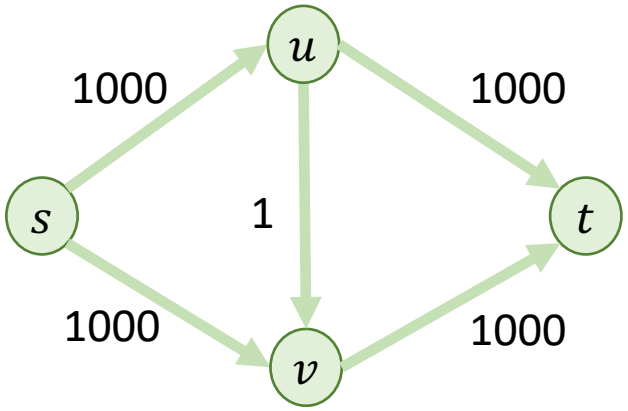
Max flow graph
Max flow value = 5.



Residual graph
Min cut is $\{s, a, b, c, d, e, f\}$ vs $\{t\}$.
Min cut value = 5.

Efficiently Finding Augmenting Paths

How long will it take Ford-Fulkerson to find max flow? That may depend partly on luck…



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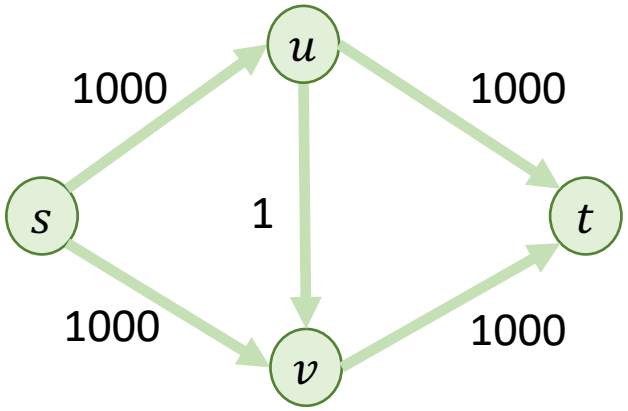
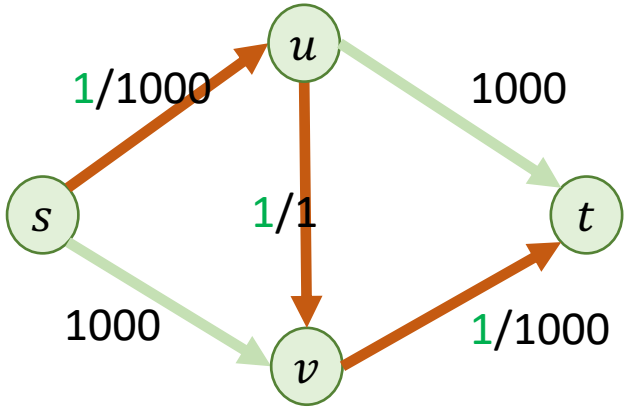
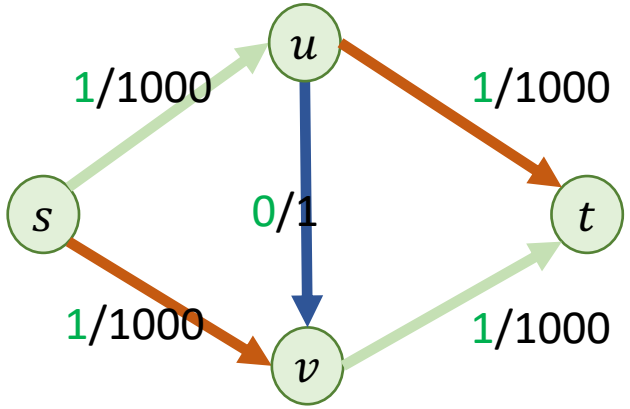
Example: In the graph below, any augmenting path crossing the **middle edge** will grow by **one unit at a time**. The **max number of augmentations** is $1000 + 1000 = 2000$. **That's a lot** for such a simple graph.







This keeps going on for 2000 steps…

Can we do anything to **guarantee better efficiency**?

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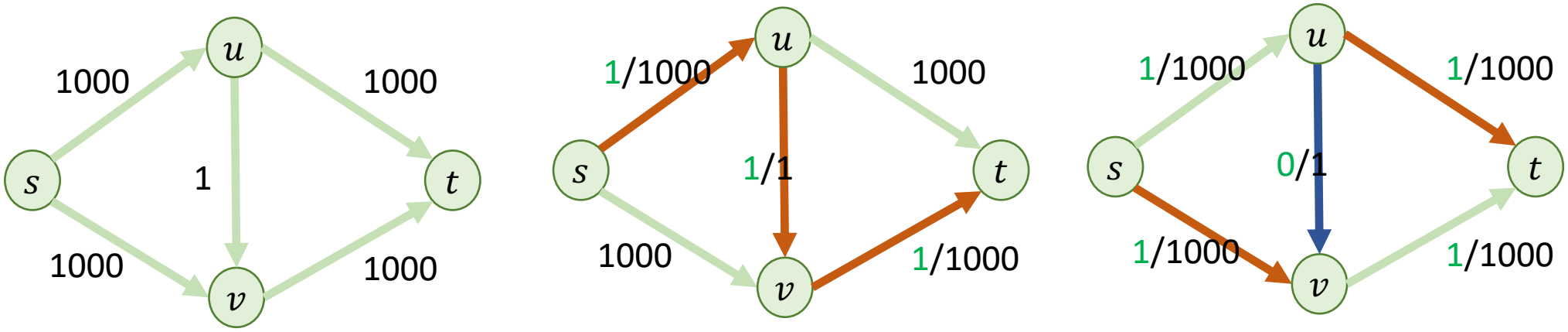


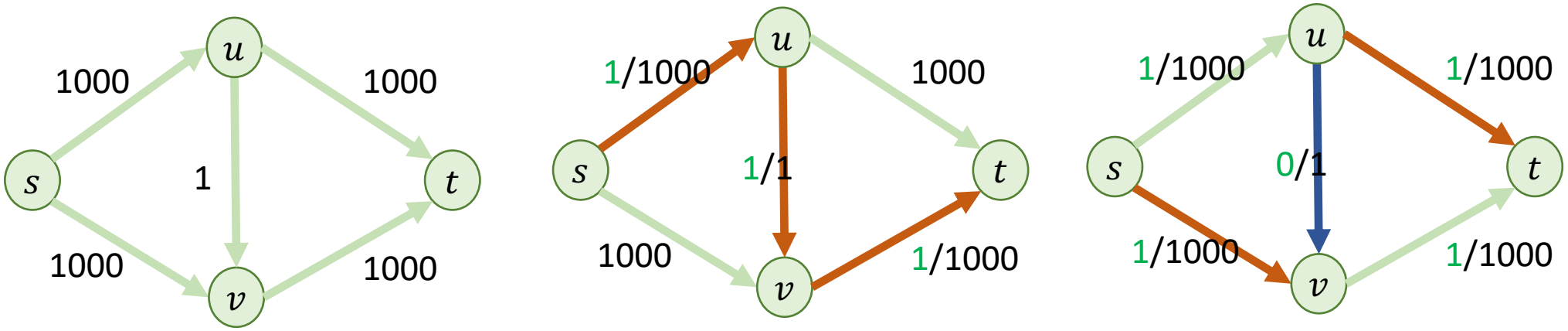
What if we know the **shortest augmenting path** from s to t?

- **Breadth first search** will find the shortest path (in terms of number of steps).
- The middle edge **cannot be part** of the shortest path from both sides – at most one of the two is shortest.
- If we find augmenting path using BFS, we are **guaranteed to not reuse** the same edge in this way.

Edmonds-Karp Runtime

If we use BFS to find augmenting paths for the Ford Fulkerson method, we call it the *Edmonds-Karp Algorithm*.

Any **iteration of Edmonds-Karp** will perform a BFS and update the flow in $\boldsymbol{O(m)}$ **time**.



Every iteration of Edmonds-Karp will **fully saturate** at least one of m edges.

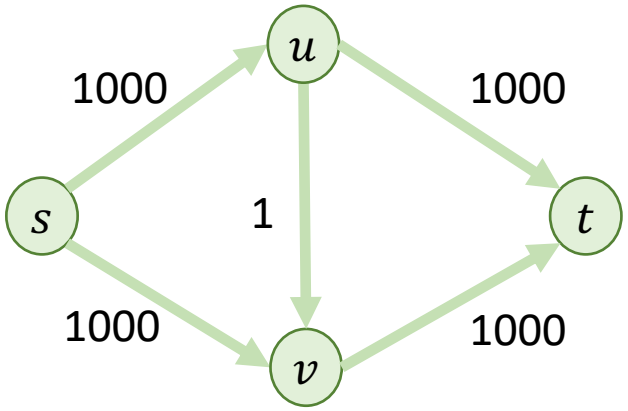
- If two augmenting paths are the **same length**, one will **not undo** an edge of the other.
- There are a total of $\boldsymbol{n-1}$ **possible path lengths**, which BFS considers in order.

Result: $O(nm^2)$ runtime.

Another Efficient Approach: Capacity Scaling

Premise: Look for higher-capacity augmenting paths first.

- Pick a **high value Δ**. Only look for paths whose **capacity is $\geq \Delta$**.
- If there are no such paths, **make Δ smaller** and try again.
- By the time $\Delta = 1$, we will find **all possible augmenting paths**.



Proposal: Make Δ a **power of 2**, and **divide by 2** when necessary.

- We can **compose any number** as a sum of powers of 2.
- The initial Δ can be the **largest power of 2** that is $\leq$ some edge.

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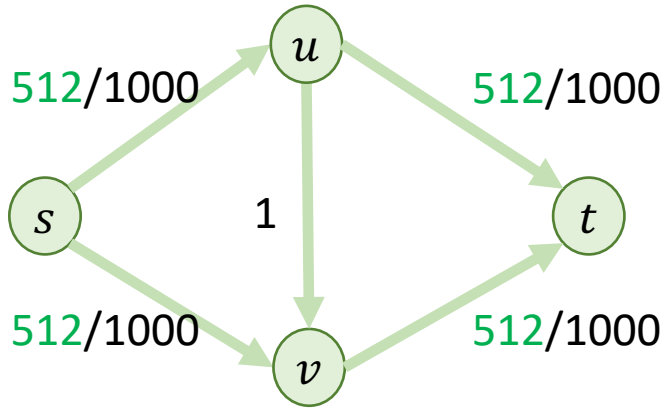

$\Delta = 512$

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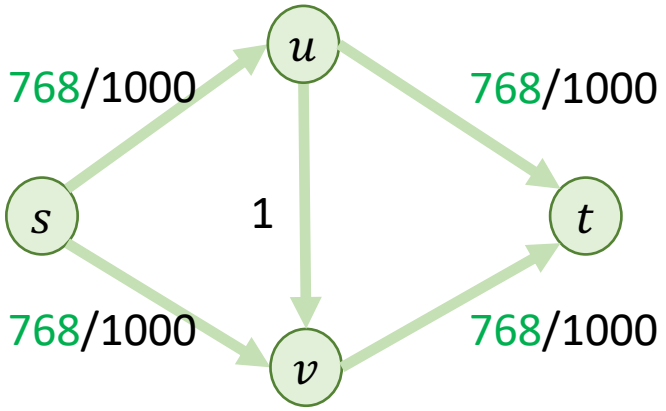

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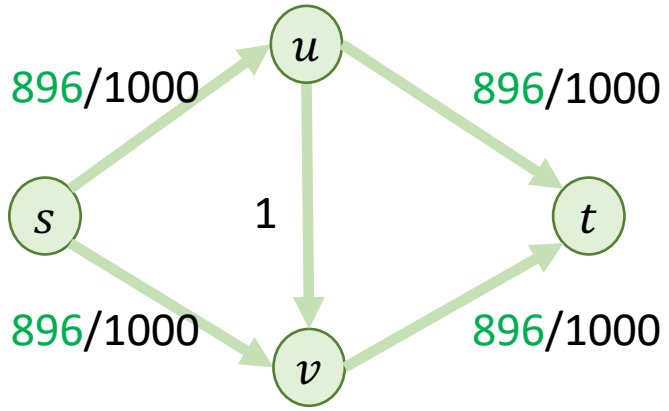

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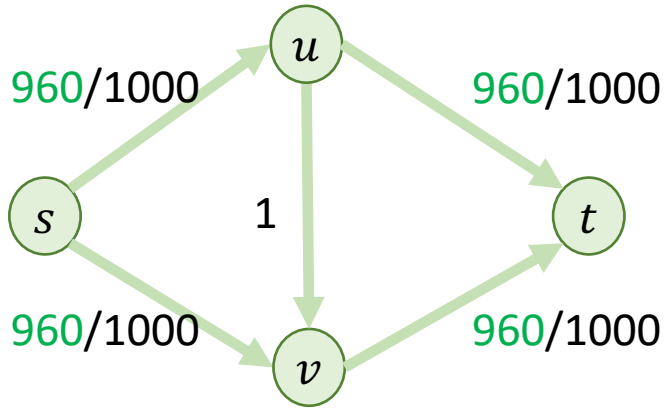

$\Delta = 64$

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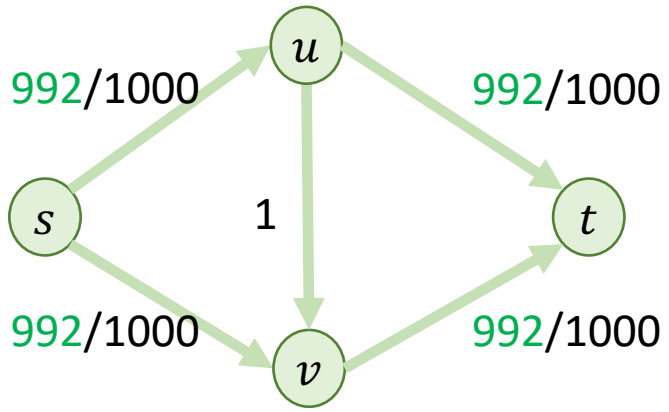

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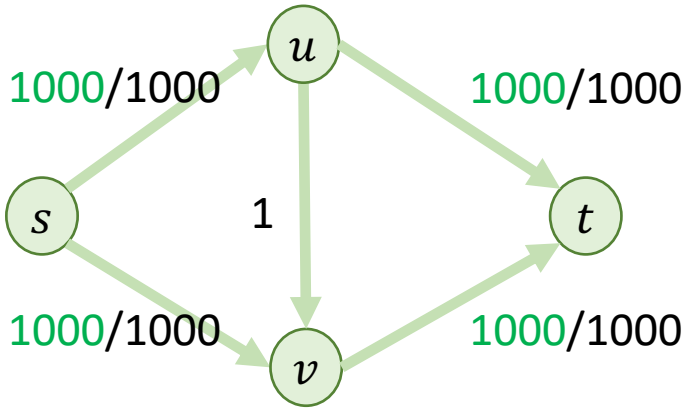


$\Delta = 8$

Proposal: Make Δ a **power of 2**, and **divide by 2** when necessary.

- We can **compose any number** as a sum of powers of 2.
- The initial Δ can be the **largest power of 2** that is $\leq$ source capacity.

Runtime:

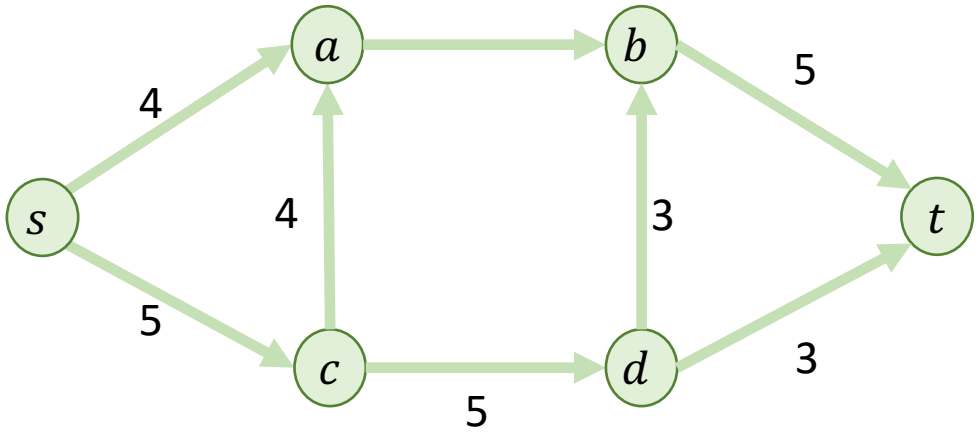
- As before, each **augmentation is $O(m)$** due to search costs.
- For each Δ, at most m **augmentations** can happen.
  - Each edge can be increased at most once if we use BFS.
- There are $O(lg\ C)$ **values of Δ** to check, where C is the capacity.

Overall: $O(m^2 lg\ C)$

Unlimited Capacity

Some edges might not have practical limits to the flow.

- How would this be represented in a flow network?



Example: Suppose there is **no limit** to the flow **from a to b**.

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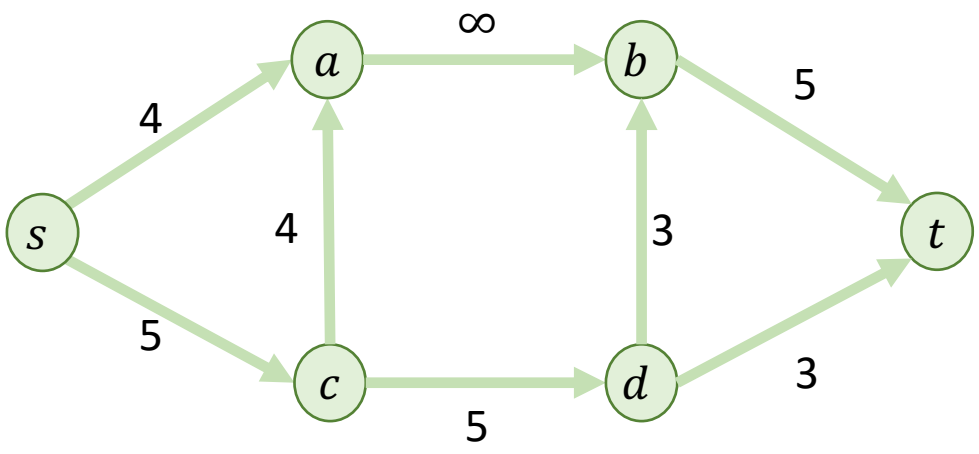


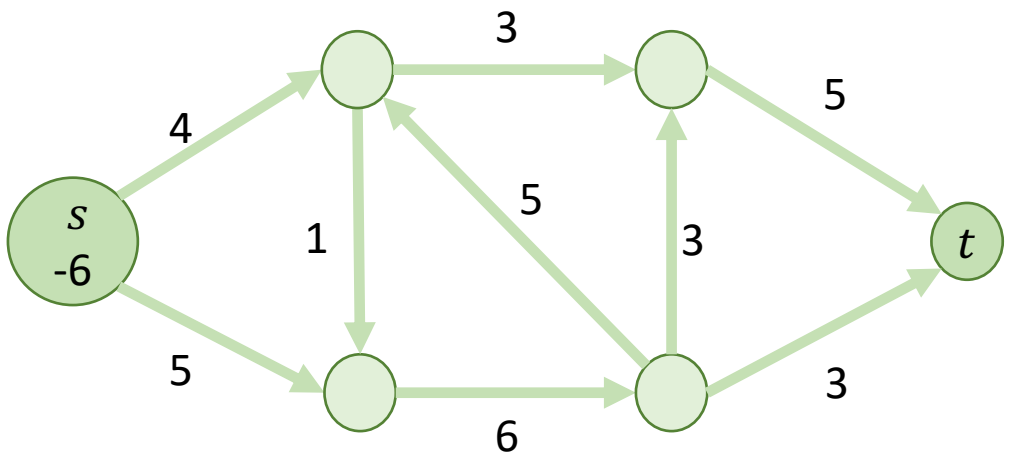
Example: Suppose there is **no limit** to the flow **from a to b**.

Solution: Simply label the capacity as **infinity**.

Producer Nodes

We may want to add simple, practical constraints to our flow network.

Imagine that our source has a **limited production capacity**.
- We could denote that with a **negative capacity value** on the source itself.



Example: The graph on the left is different from a basic flow network because it has a **source limit** of 6.

No more than 6 units of flow can be produced by the source.

We know how to solve an ordinary **max flow**.
- Can we **use what we know** to solve this problem?

Producer Nodes

We may want to add simple, practical constraints to our flow network.

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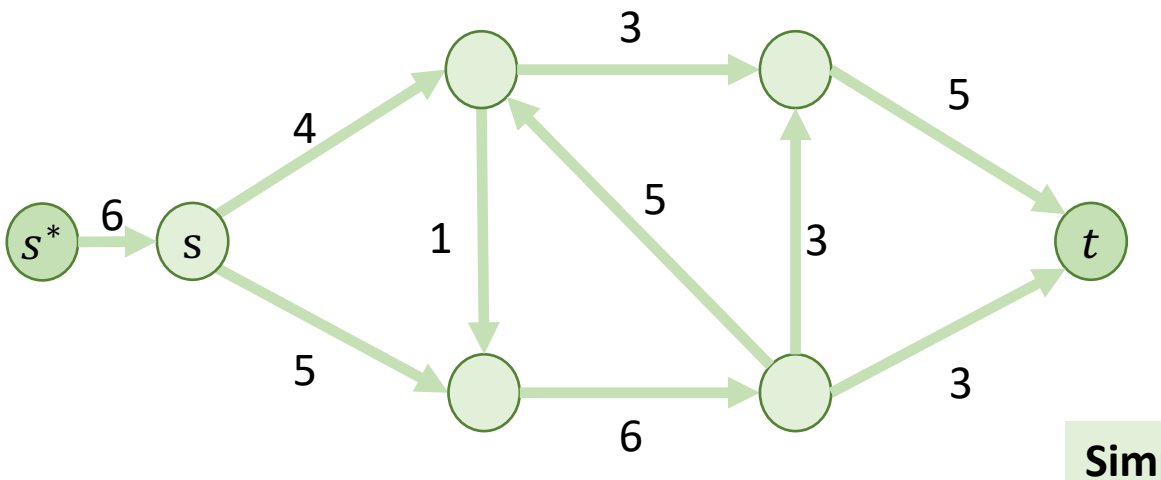


Simple solution: Create a new source.

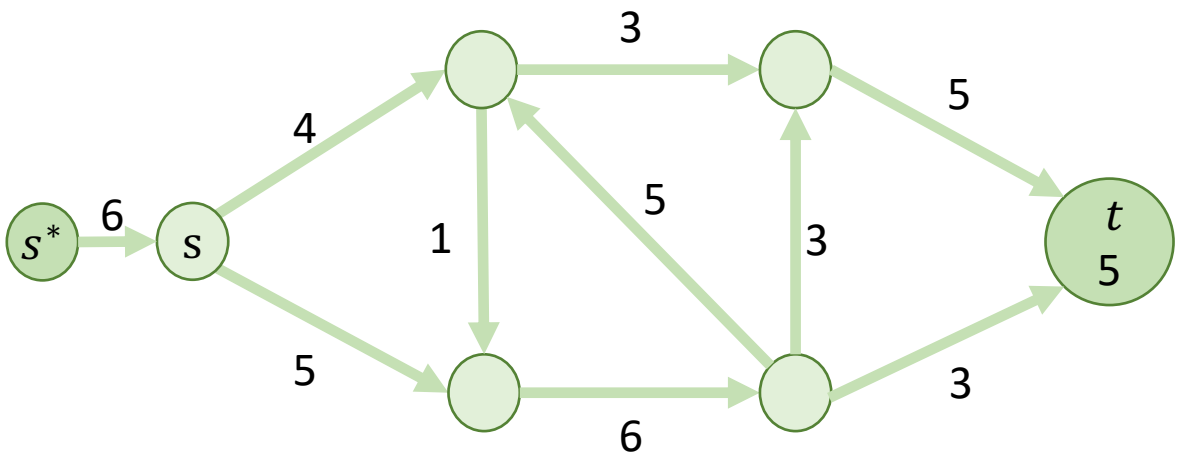
- We can only reach s from s^* by a path with **bounded capacity**.

Consumer Nodes

What if our sink has limits to what it **can consume**?.

Imagine that our sink has a **limited consumption capacity**.

- We could denote that with a **capacity value** on the sink itself.



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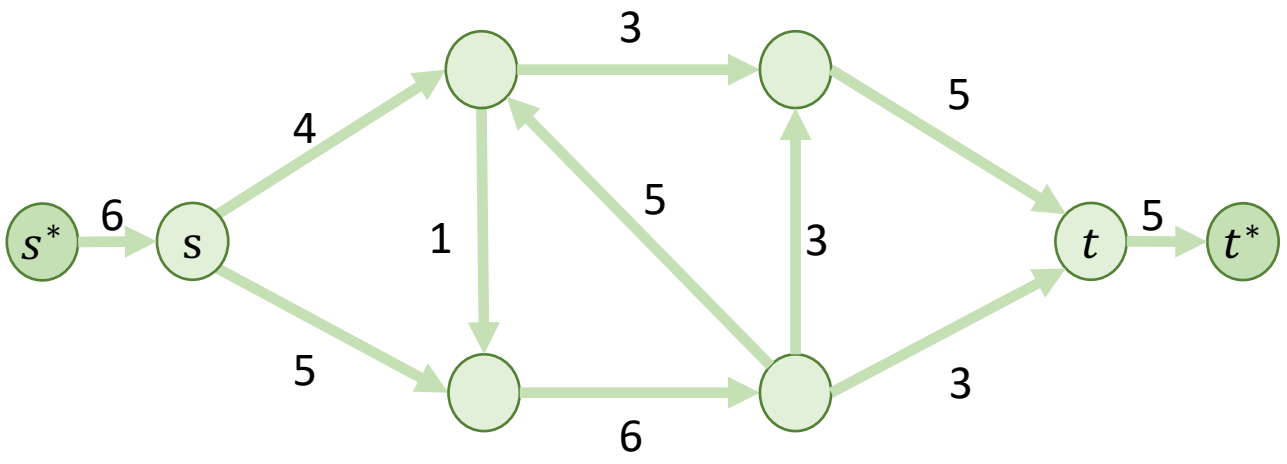


As with the source, we can restrict the sink by **creating a new sink**.

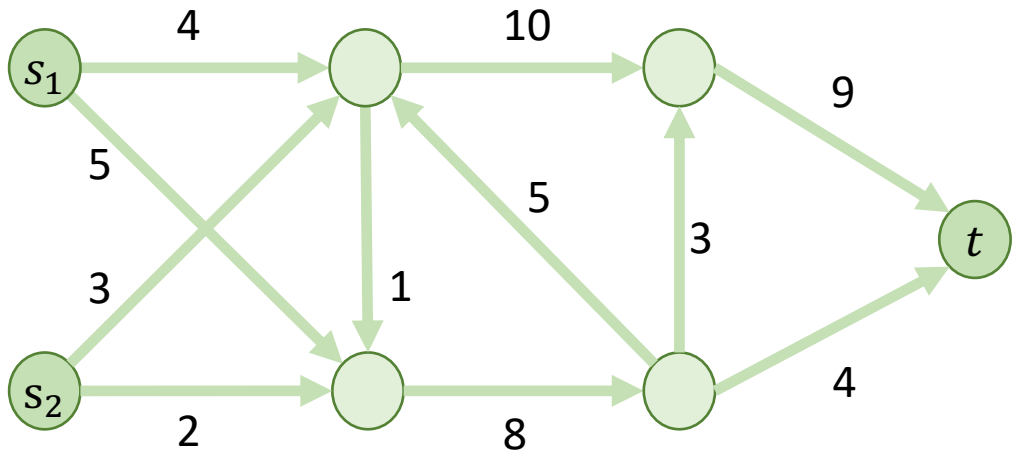
- The **only way to reach** the new sink is via a **capacity-limited edge**.

Dual Sources

Why would we need to limit capacity at both **source and sink**?

- Wouldn't the result be the **same either way**?

Ah, but consider what happens if we now had **two different sources**.



We would still like to solve this using **techniques we already know**.

Dual Sources

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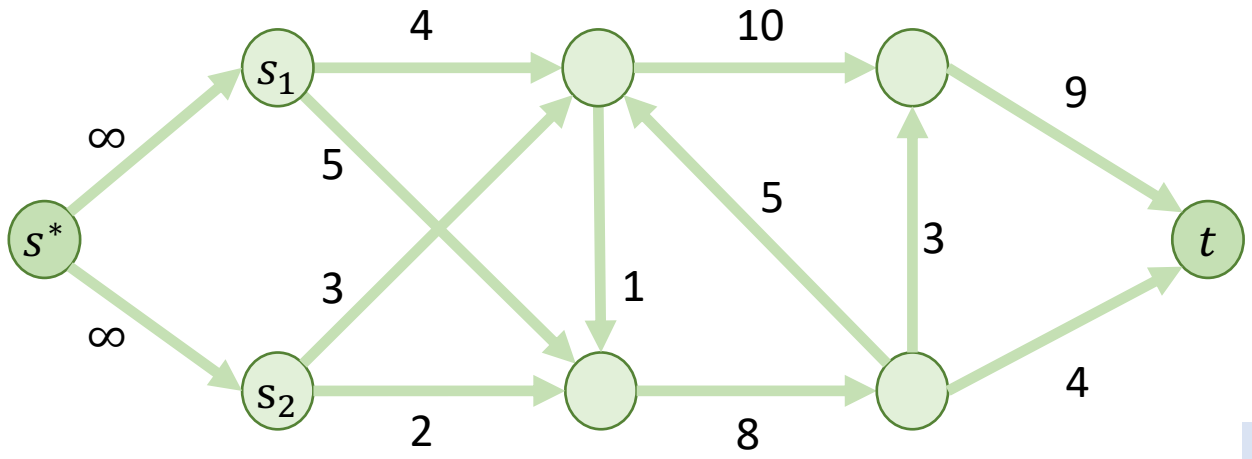


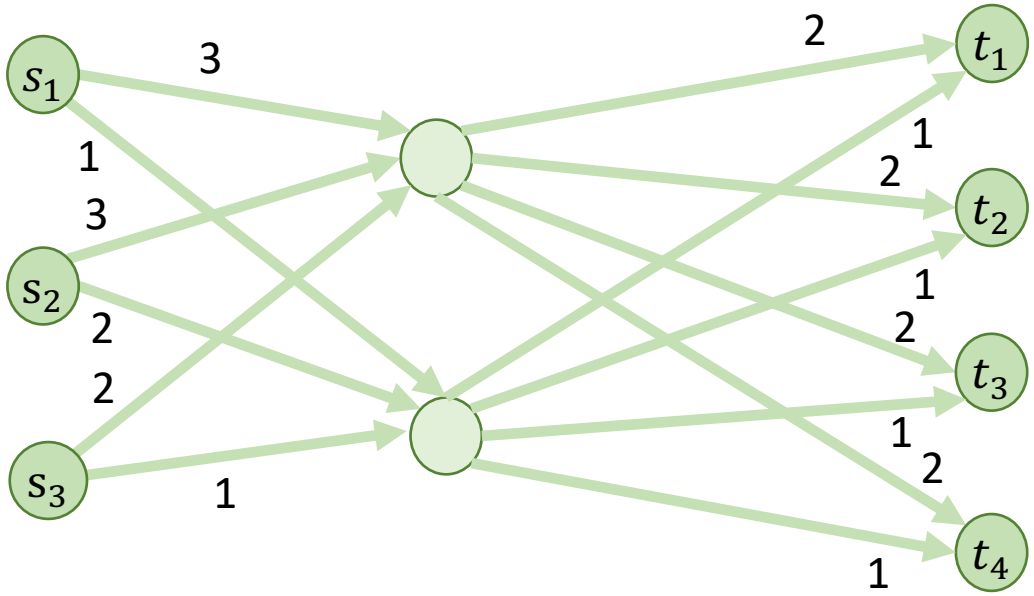
Solution: Feed **both sources** from a new **single source**.

We do not know how much flow each source will put out, so we can use **infinite capacity lines** to feed each source.

Multiple Sources and Sinks

Can we generalize situation where we have **many sources and sinks**?

Example: Three different sources and four sinks.



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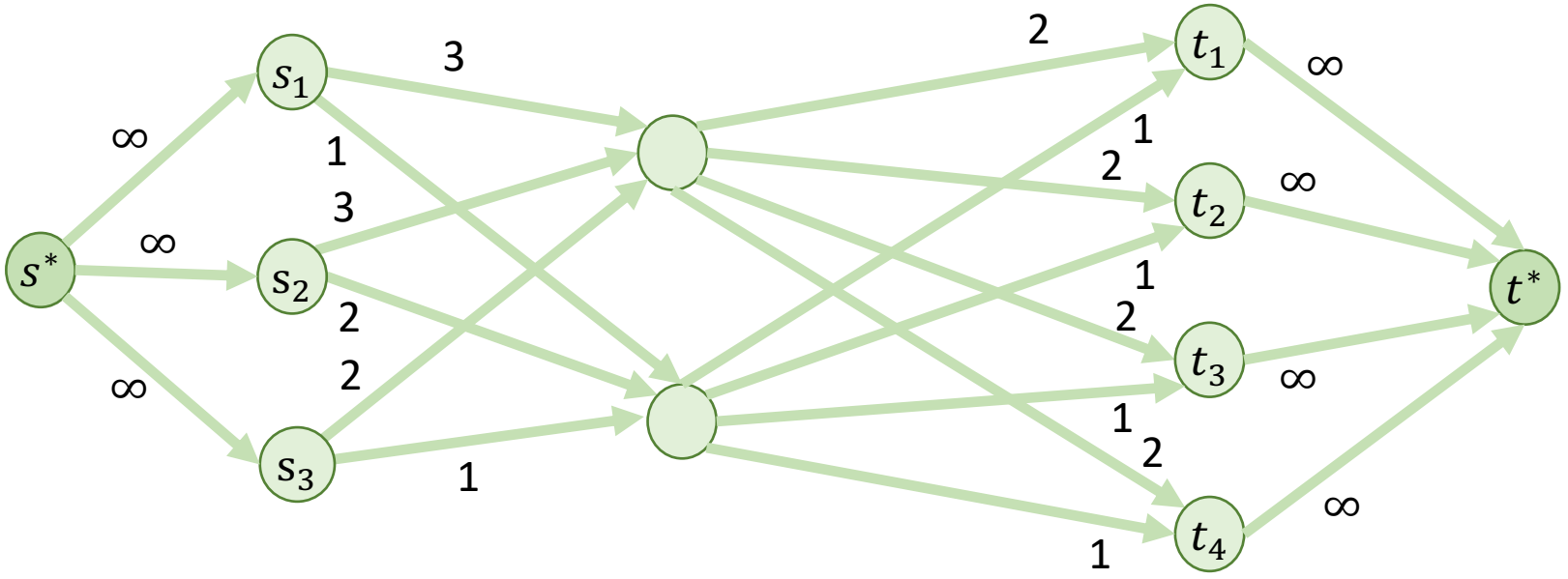
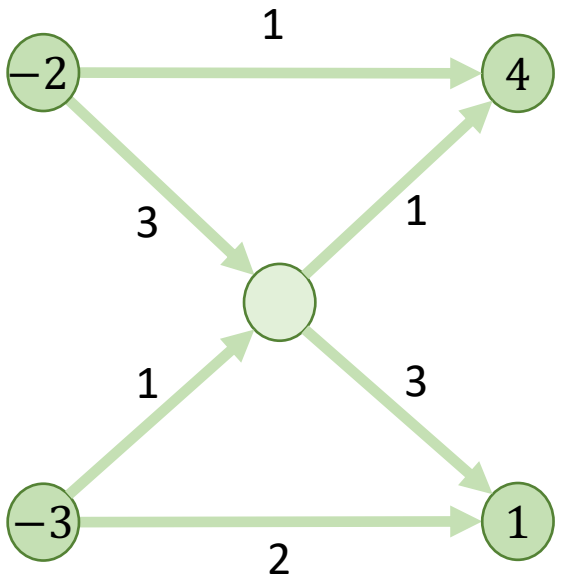


The key is to be sure we end up with **exactly one source** and **exactly one sink**.

Producers and Consumers (circulations)

Suppose we have a network in which:

- Some nodes produce flow. (negative demand, $d_v < 0$)
- Some nodes consume flow. (positive demand, $d_v > 0$)



Example: How can we arrange this graph into something we can solve as a **max flow**?

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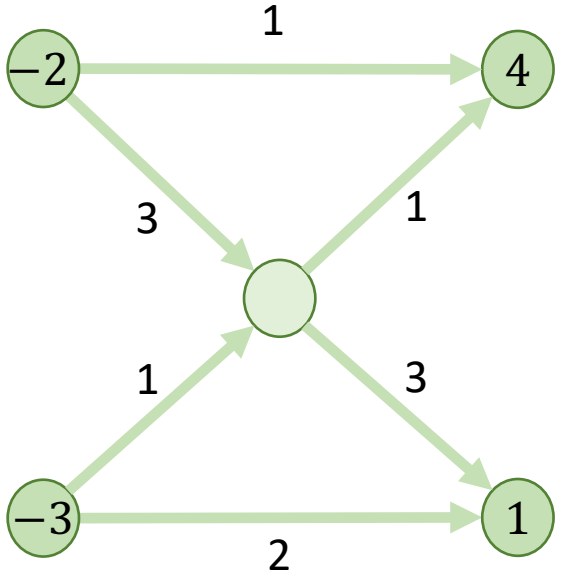

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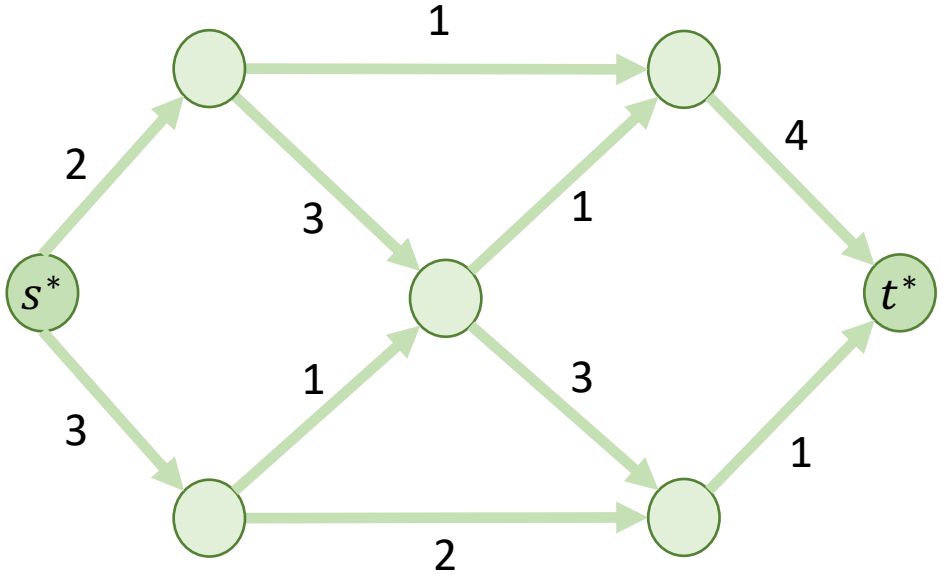
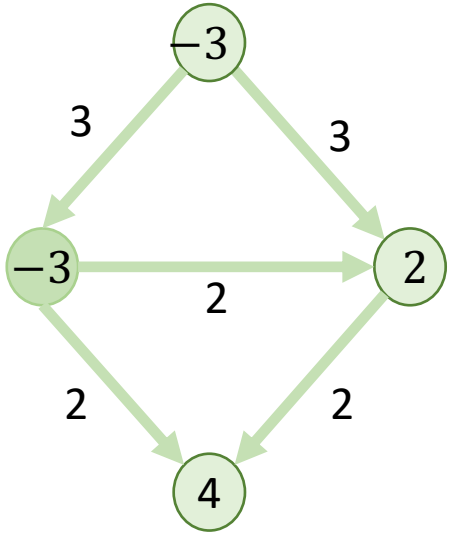


Solution:

- New source **with edges to each producer**.
  - Edges from source to producer reflects producer capacity.
- New sink **with edges from each consumer**.
  - Edges from consumer to sink reflects sink capacity.
- Feasible solution if max flow = Σ positive demands $= \Sigma$ negative demands

Producers and Consumers (circulations)

A consumer might also be an internal node



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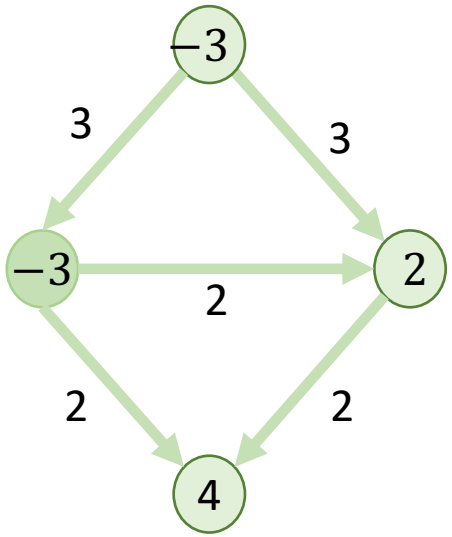

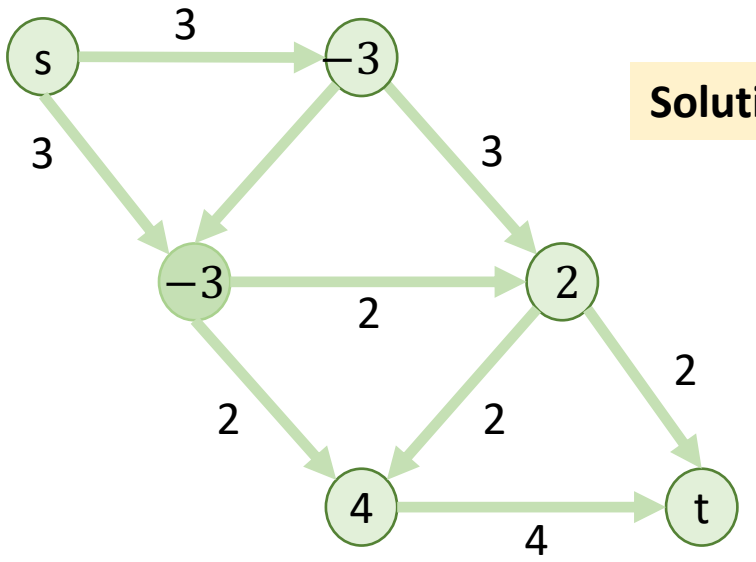

Solution: Find a max flow of 6.

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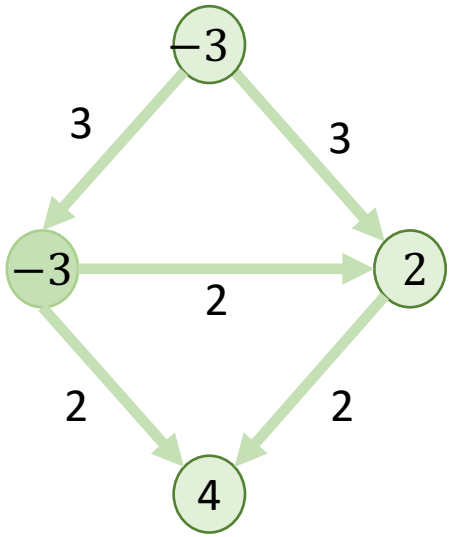

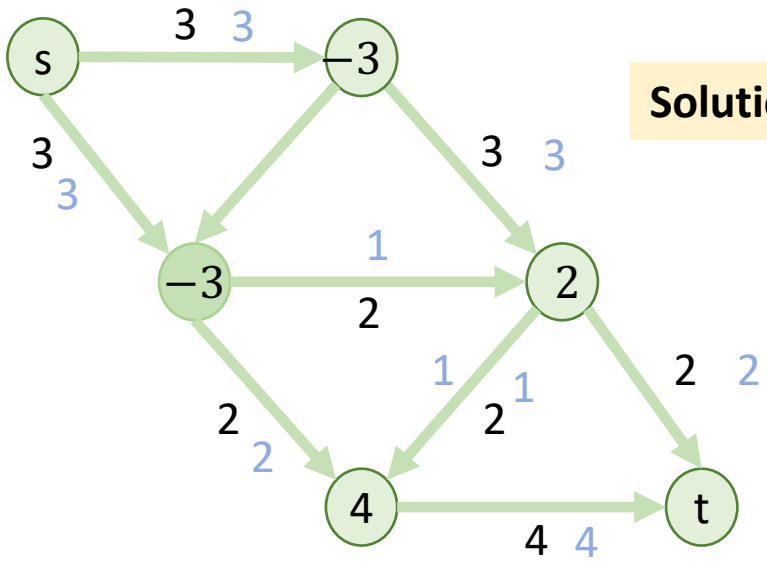

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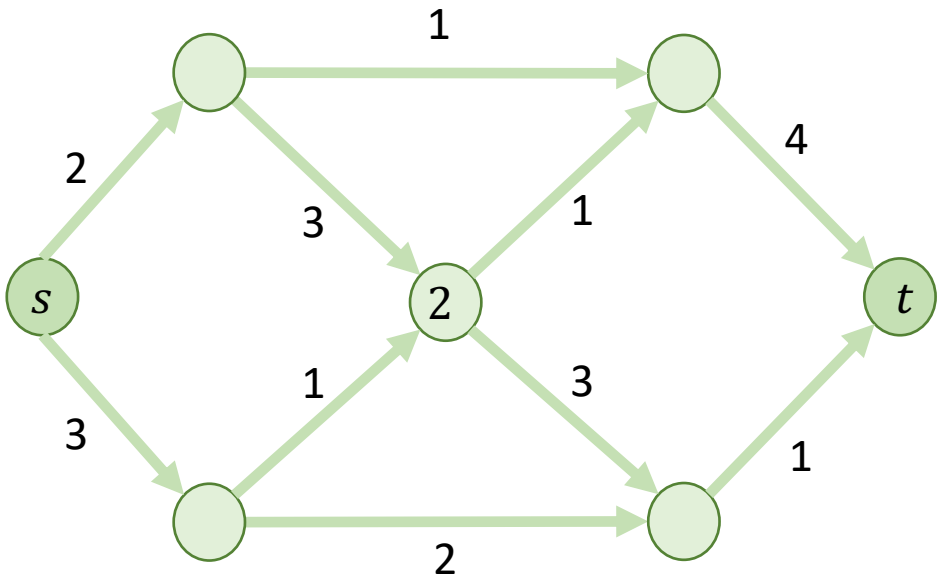
Limited Capacity Nodes

Suppose that some node by itself has capacity limits?:

Example: Suppose that only 2 units of flow can flow across the middle.



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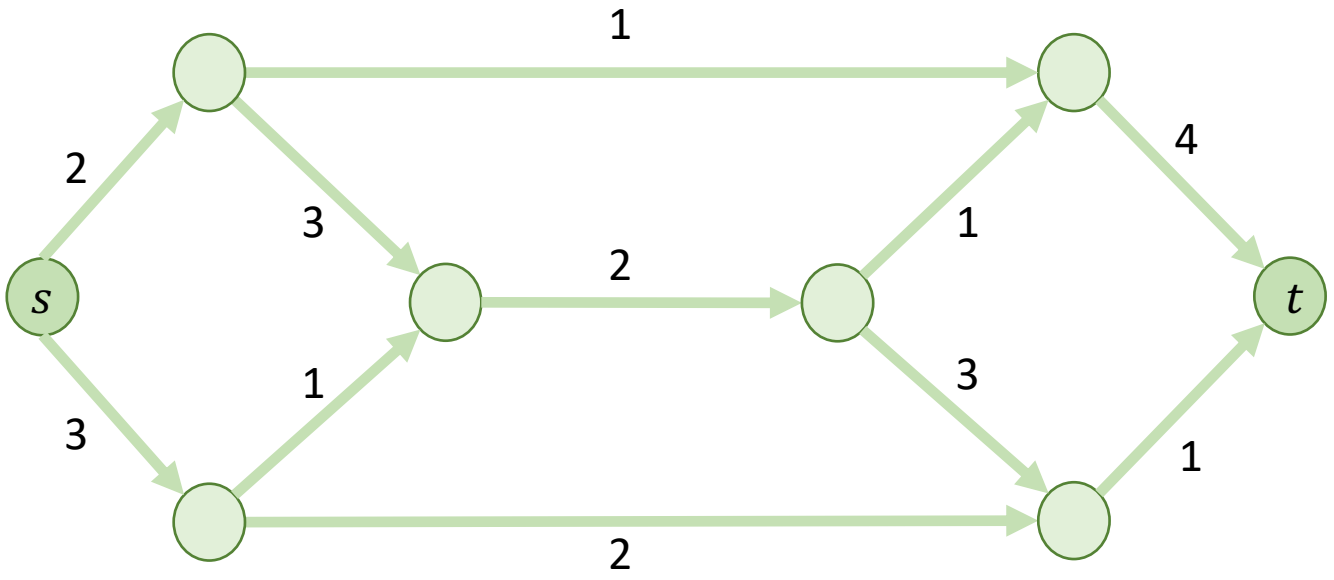


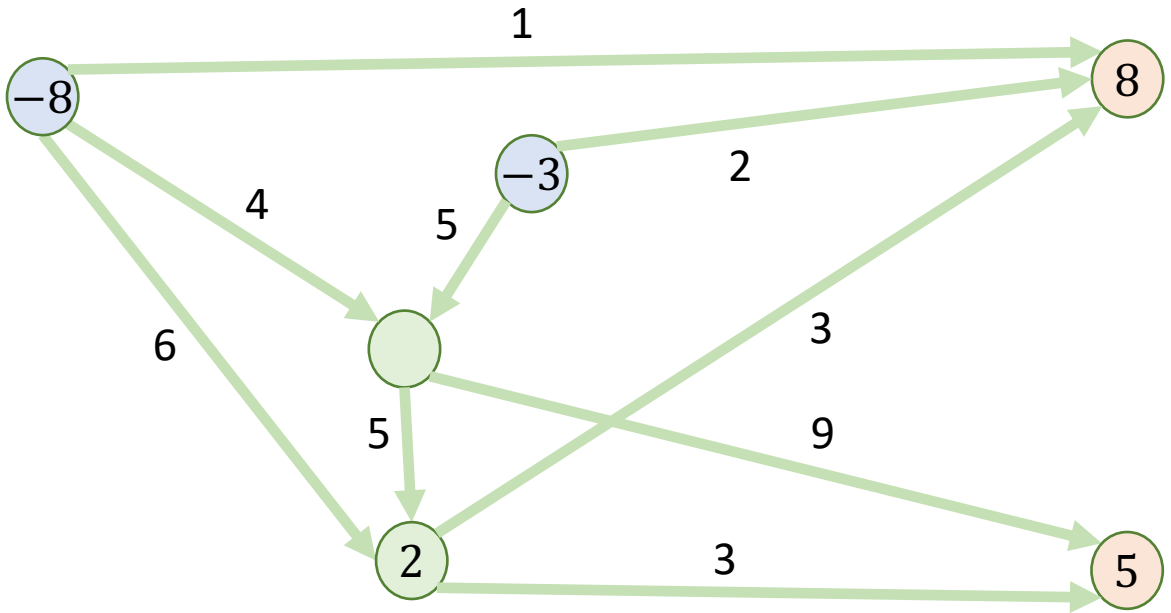
Solution: Turn **one node into two** connected by an edge. Let the edge represent the **capacity limit**.

Practice

Rewrite the following graph as a standard flow network.

Find the **max flow**.

The blue nodes are capacity-limited **producers**.
The brown nodes are **consumers**.



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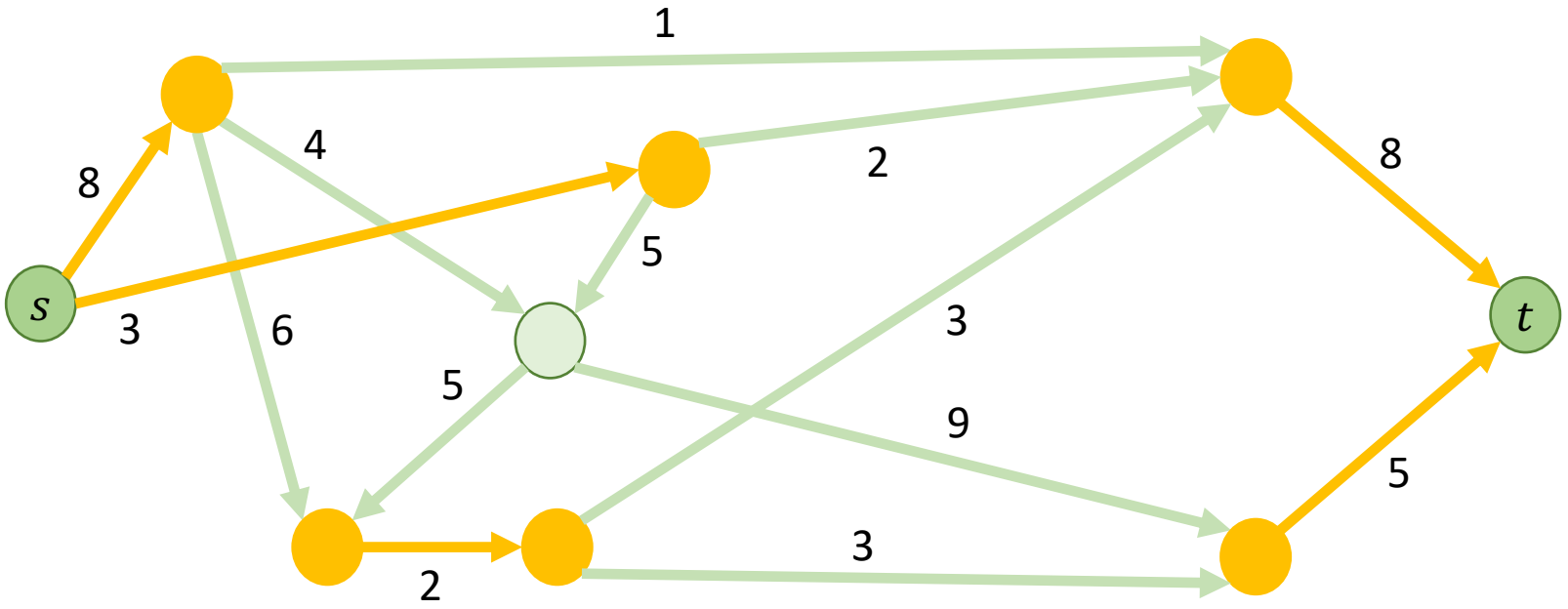

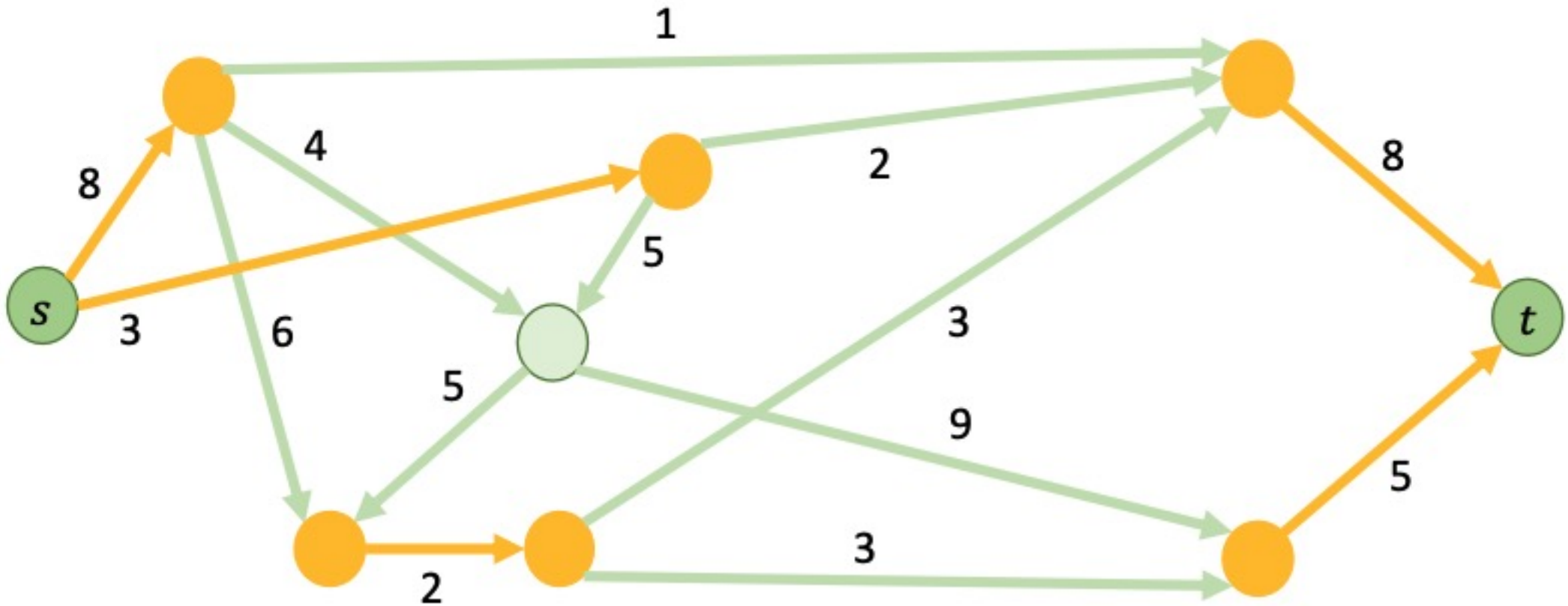
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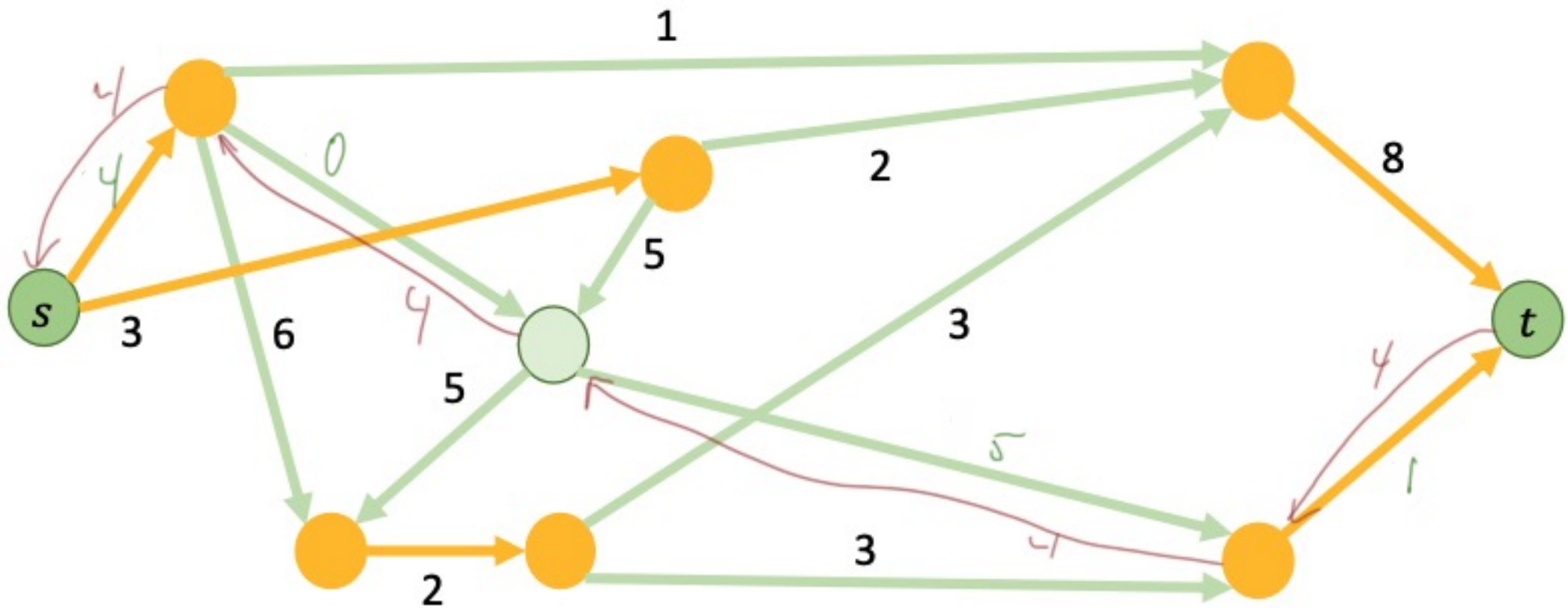
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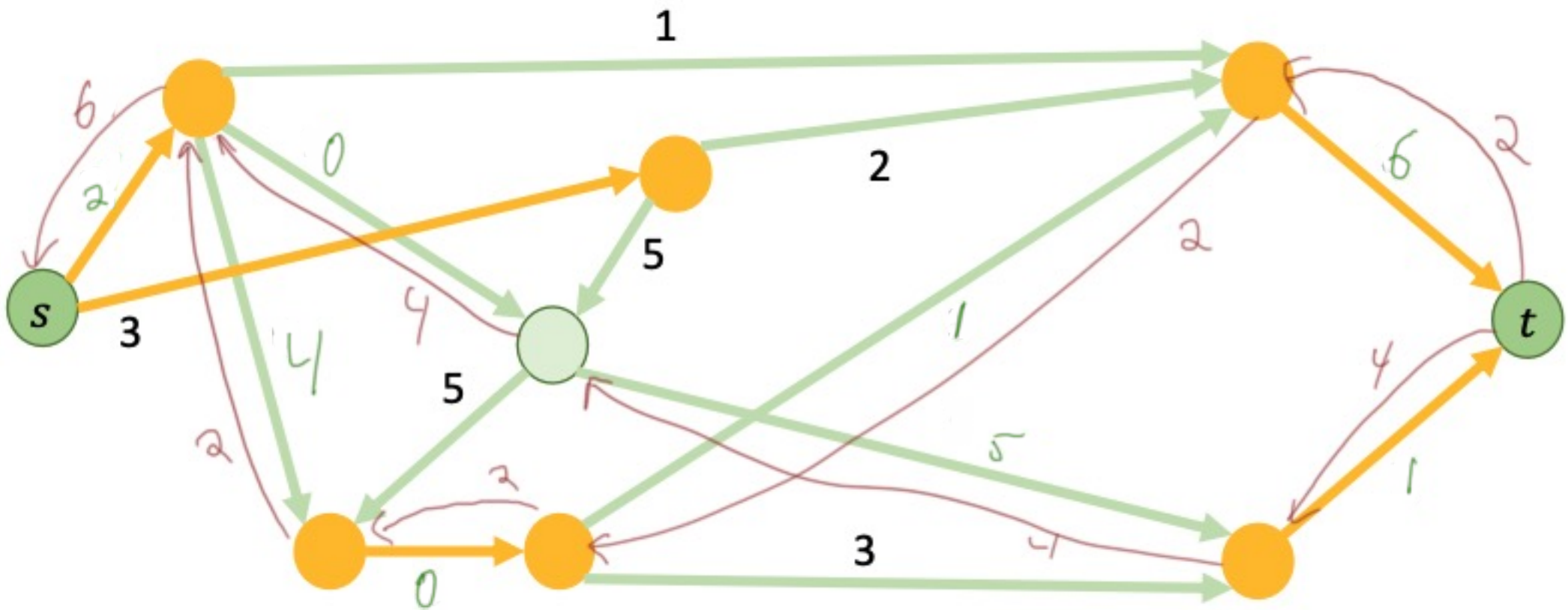
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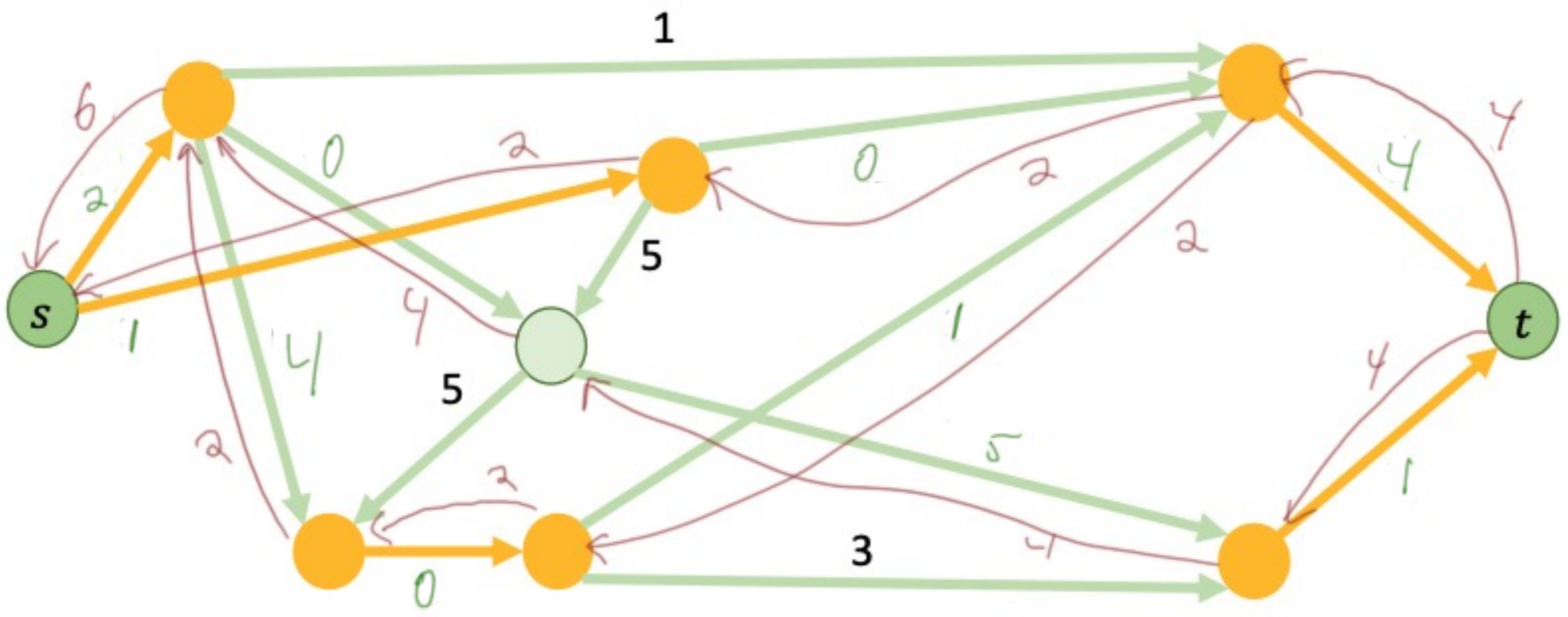

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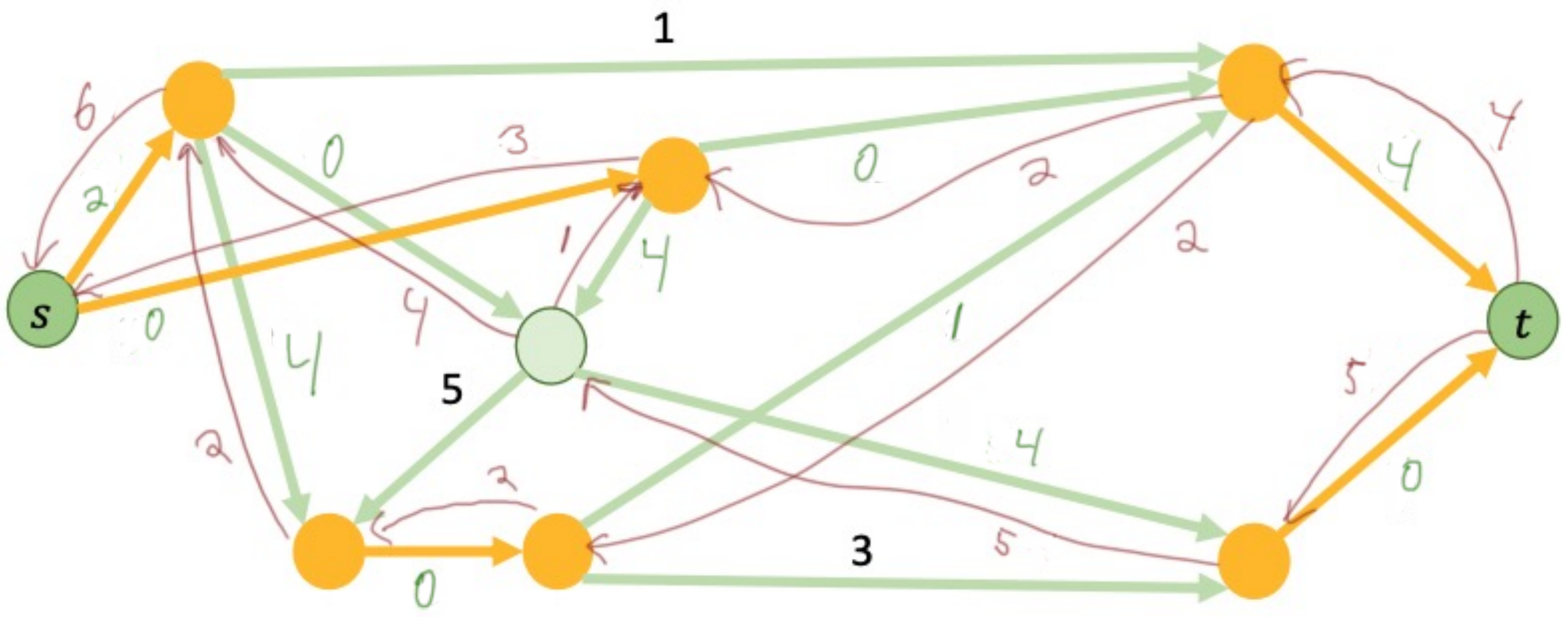
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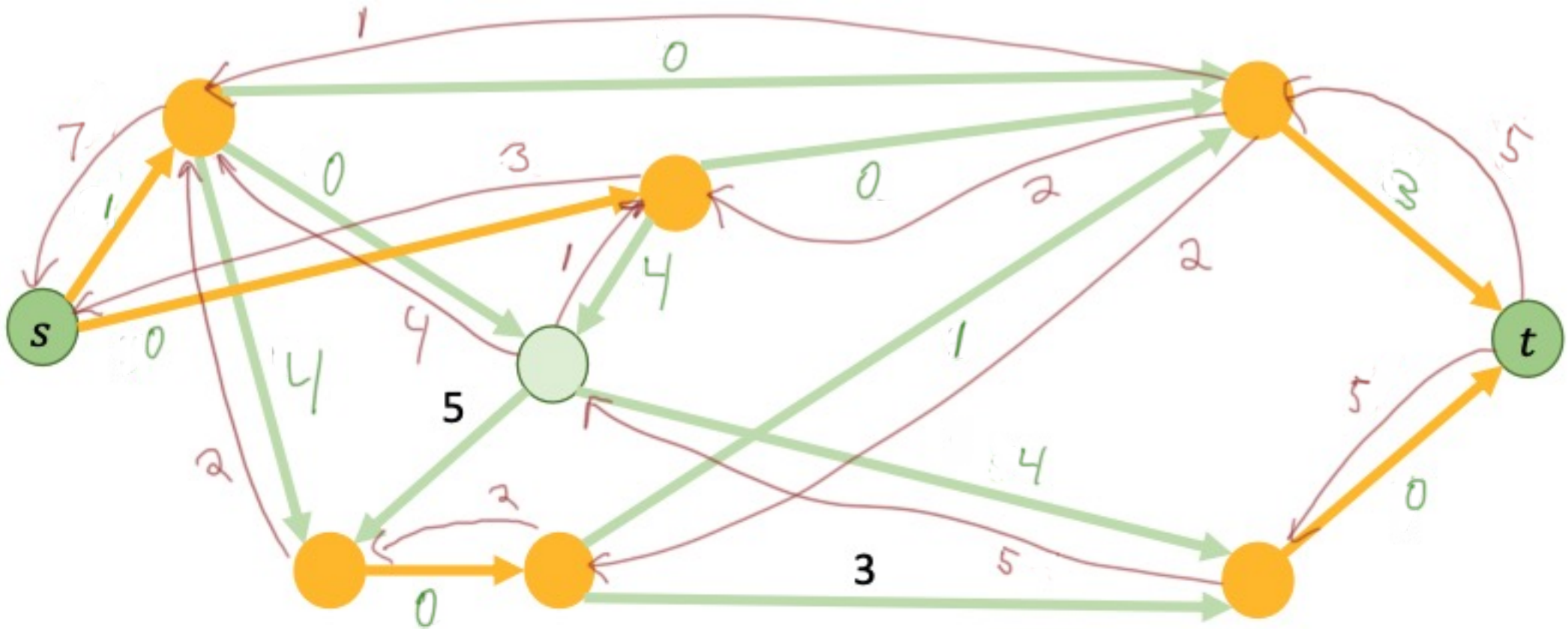
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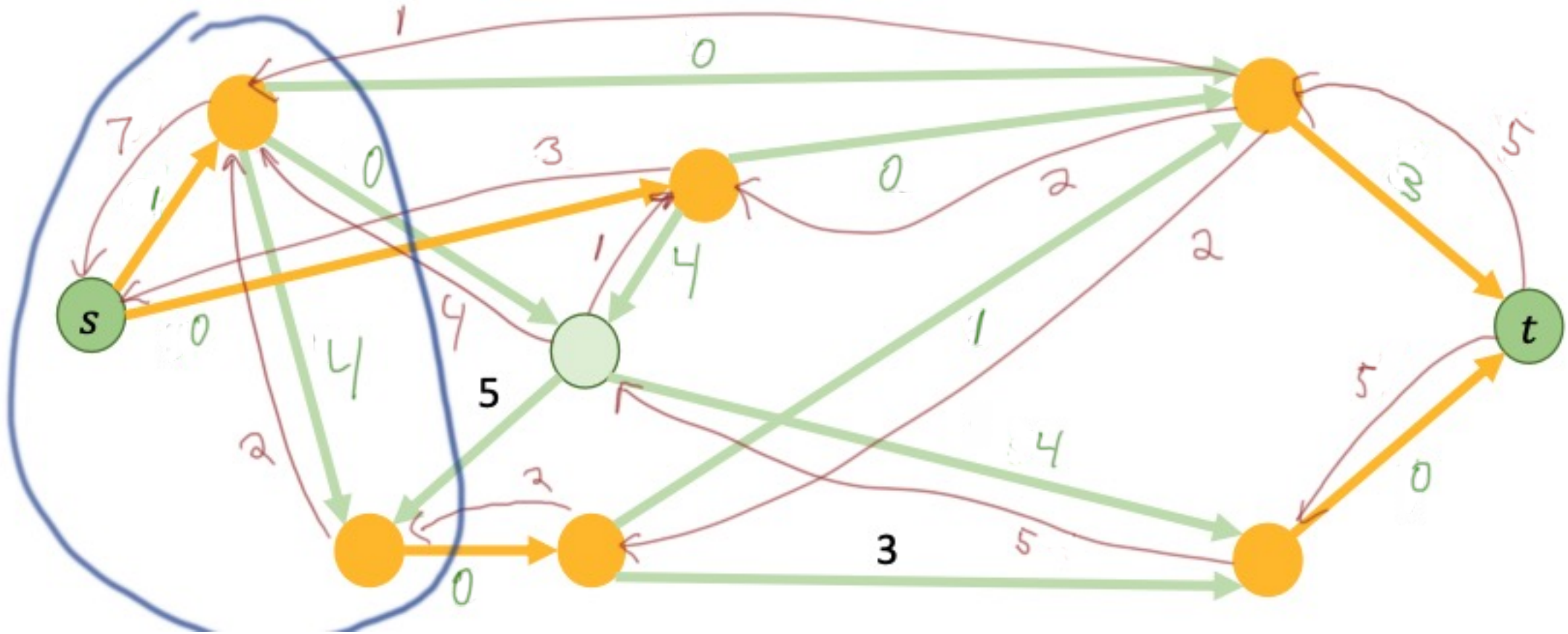
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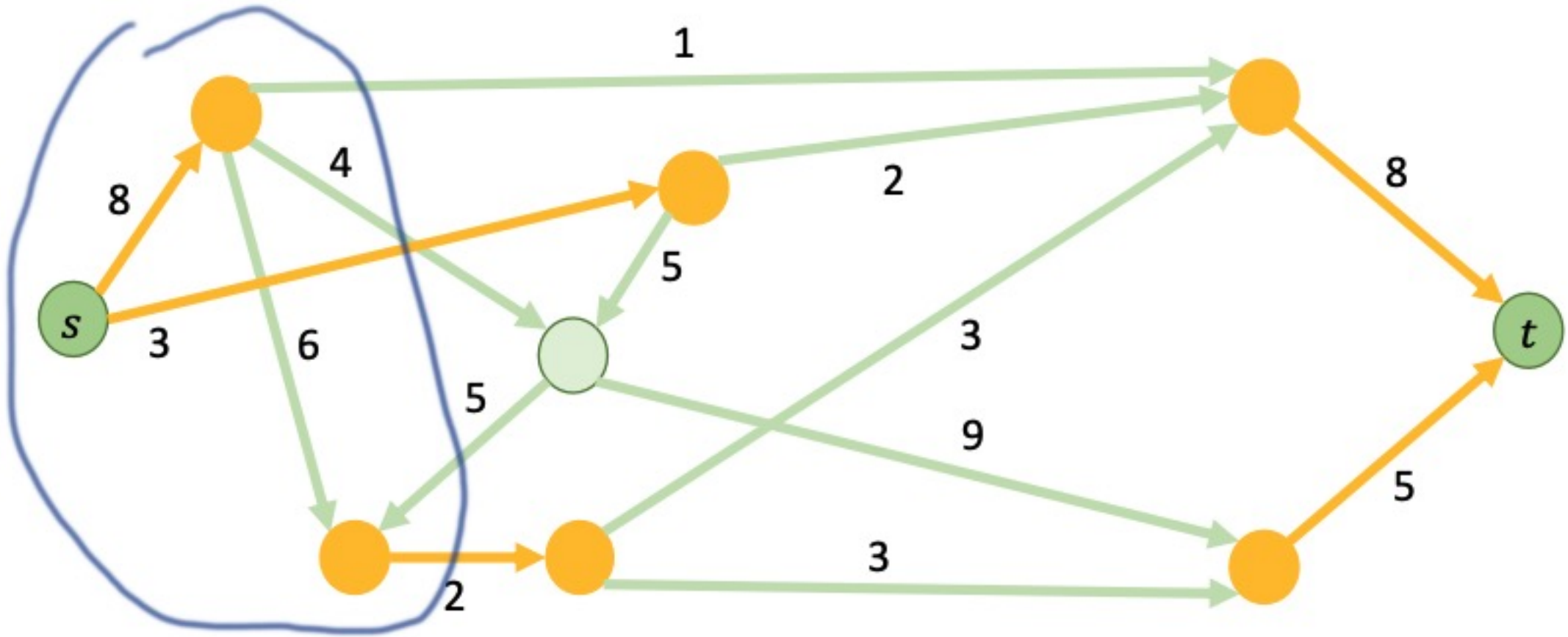
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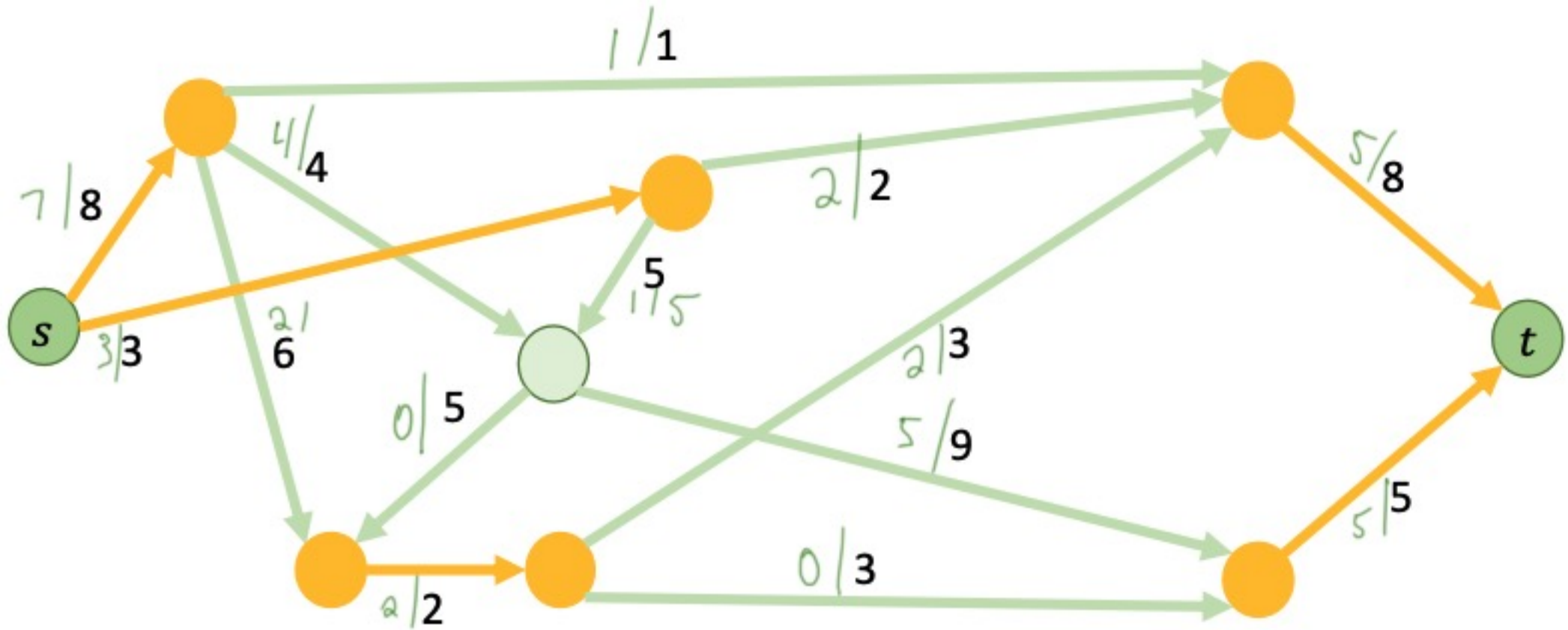
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Flow Graphs for Task Assignment

Let’s consider a practical use of max flow problems.

Goal: We want to **assign workers to various tasks**.

- Everyone can be assigned a maximum (for example, 2) tasks.
- Some people are capable of performing more than one type of task.
- Not everybody knows how to do every kind of task.

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We can represent this relationship with a **bipartite graph**.

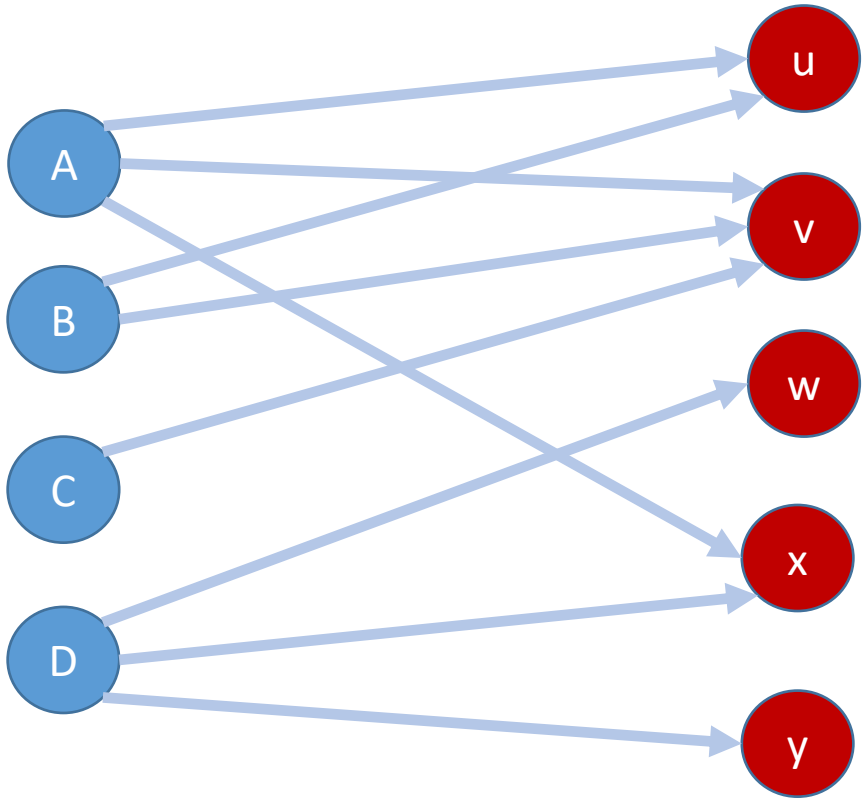
A bipartite graph is a graph which has **two groups** of nodes.

- Edges can **only cross** from one group to the other, not two nodes in one group.
- **Example**: One group can be a group of workers, the other a group of tasks.

Task Assignment, Example

The edges show **who can do which tasks**.

Workers



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Goal: Find the **most tasks** that can get done.

Hint: this can easily be turned into a **flow problem**.

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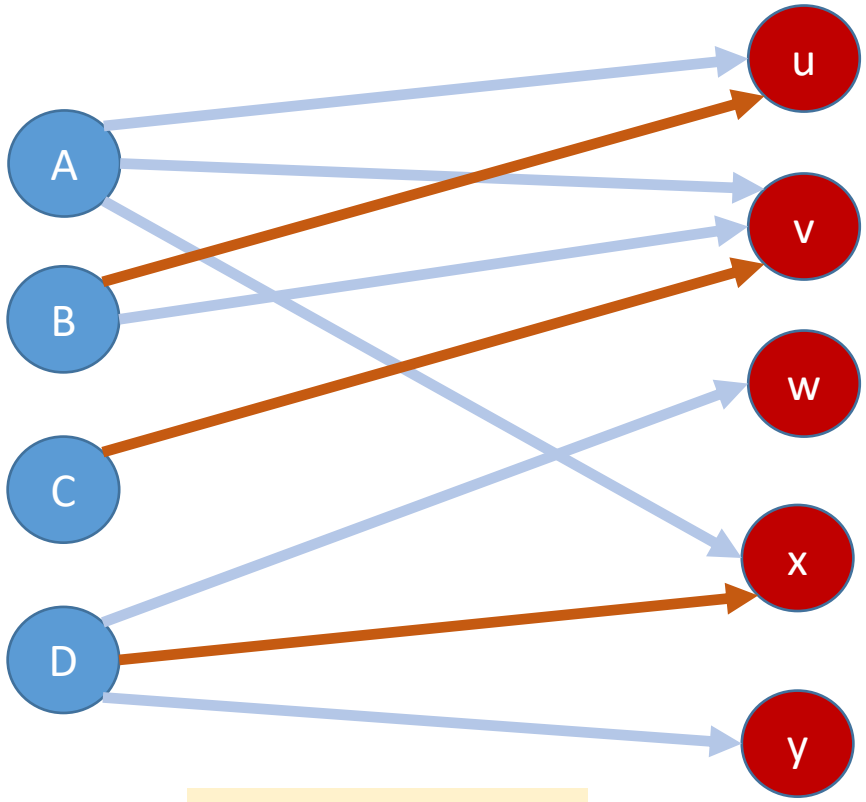

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Sample solution
Can we do better?

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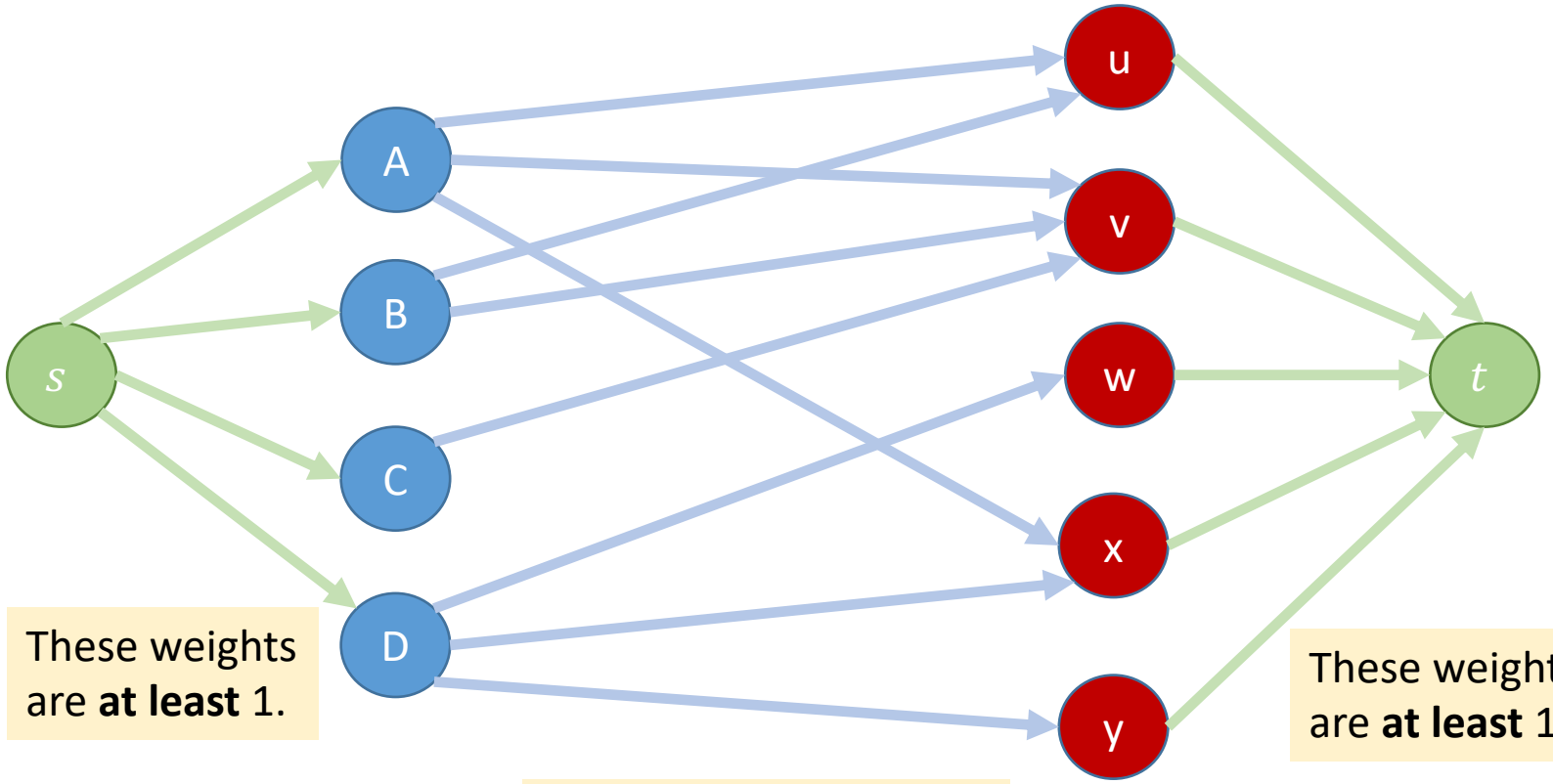

What would it mean if we use weights which are > 1?

These weights are **at least** 1.

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These weights are all 1.

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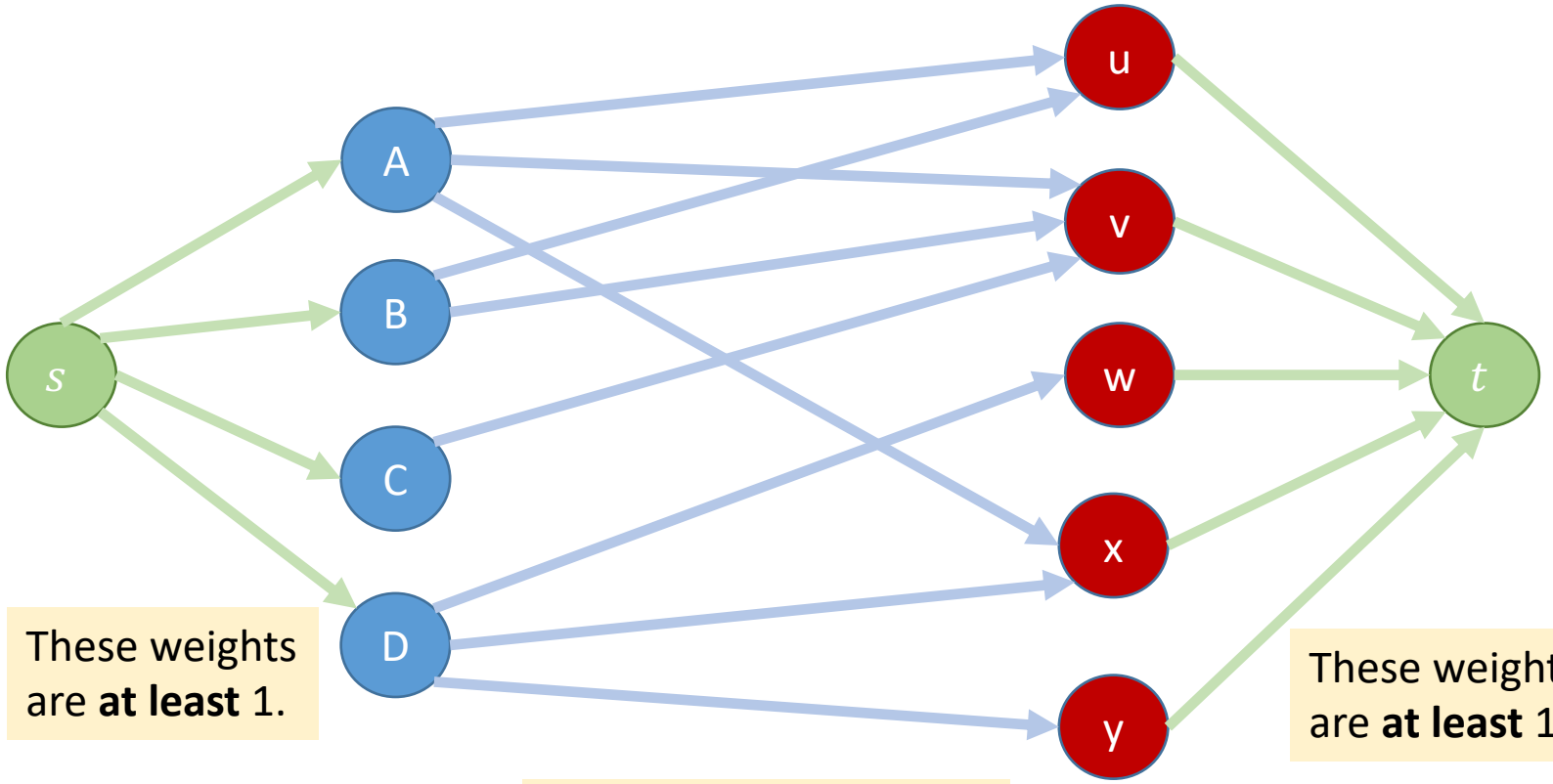

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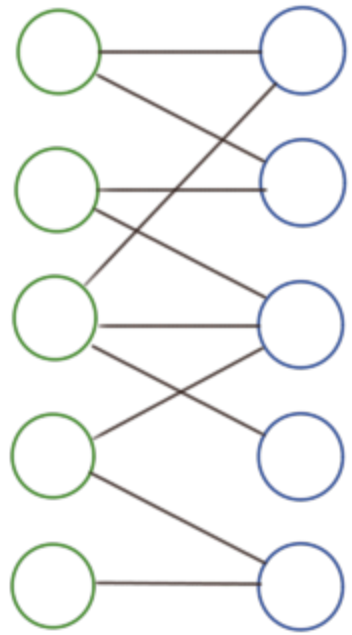
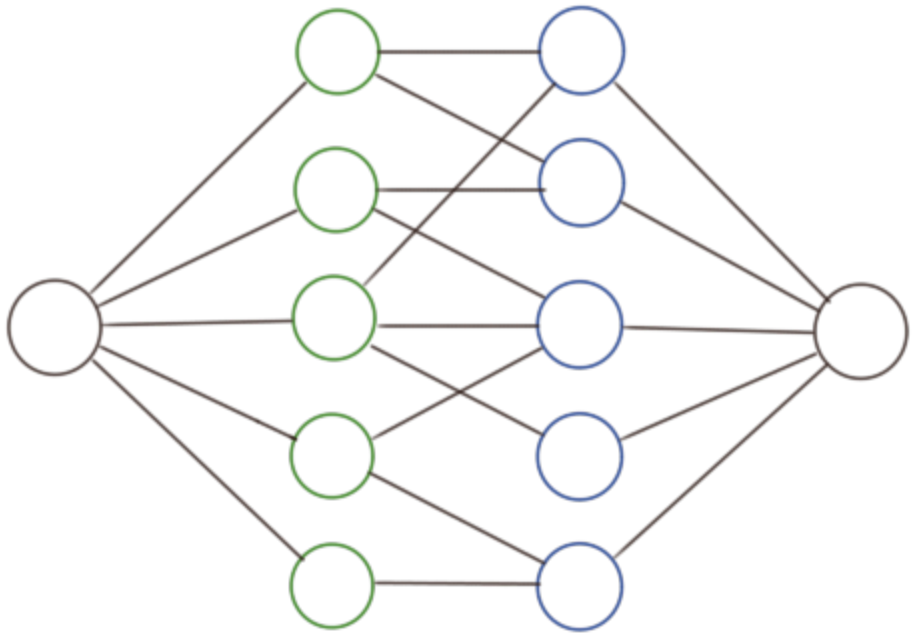
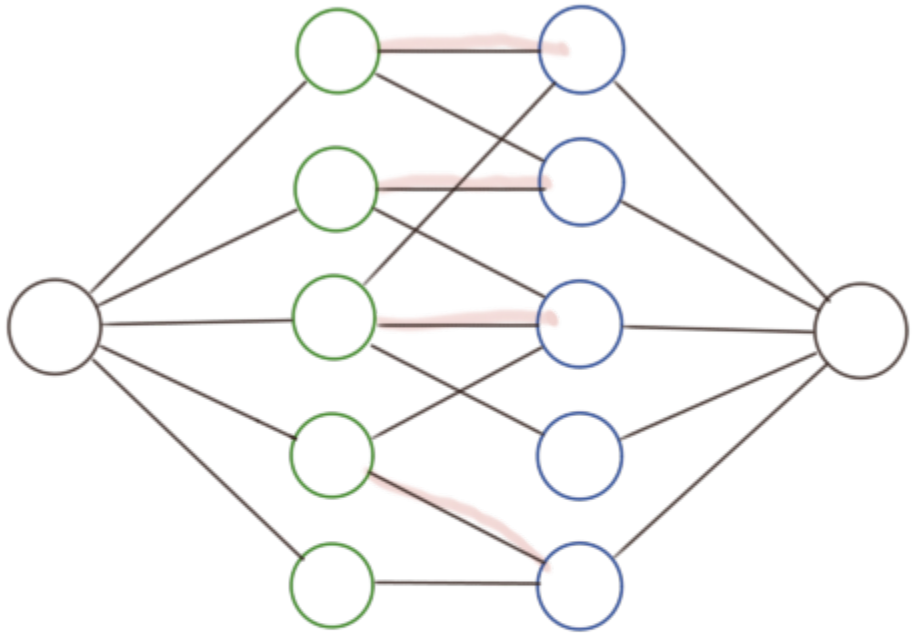
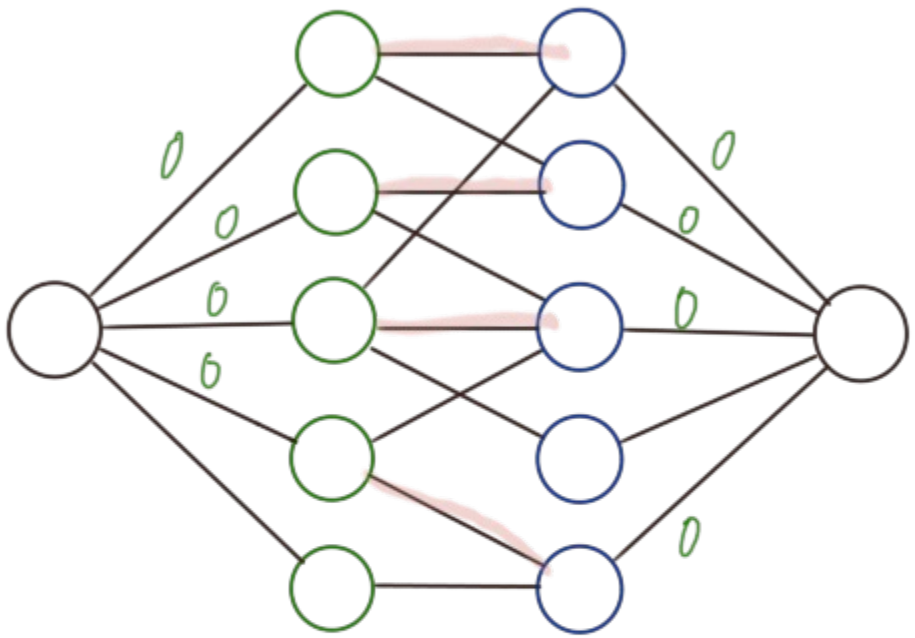
What would it mean if we use weights which are > 1?

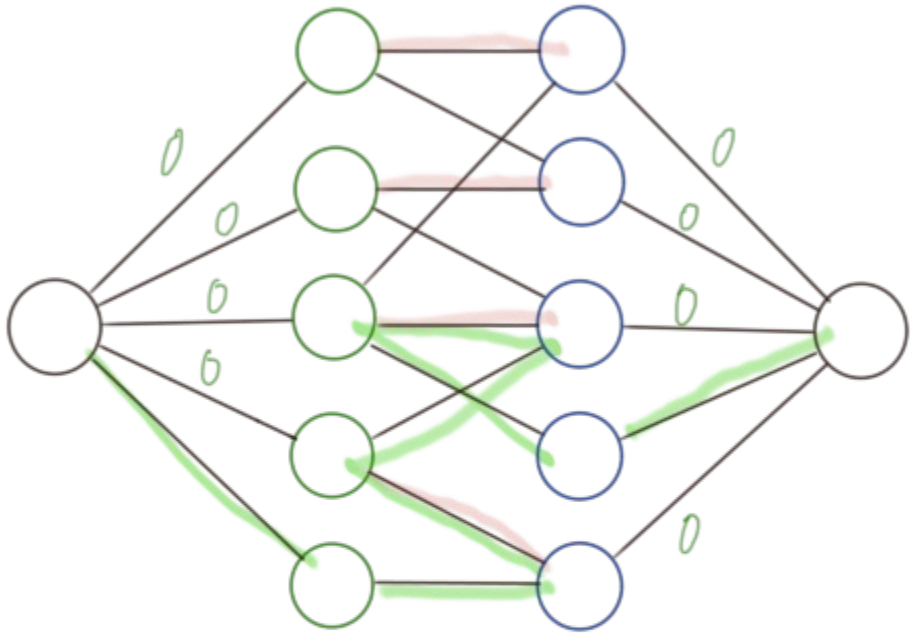
- If a worker can be assigned **more than one task** at once.
- If a task can be assigned to **multiple people** at once.

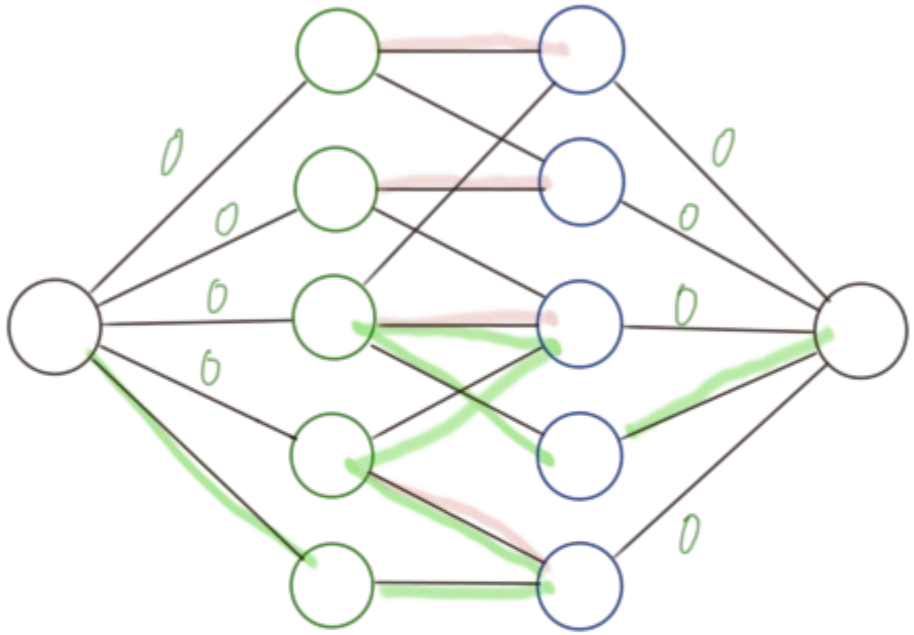








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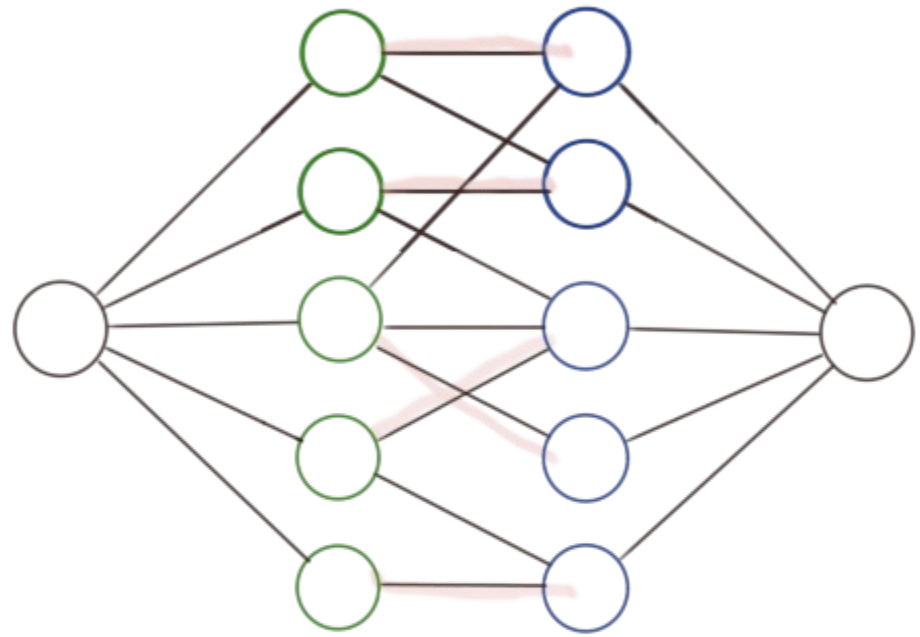

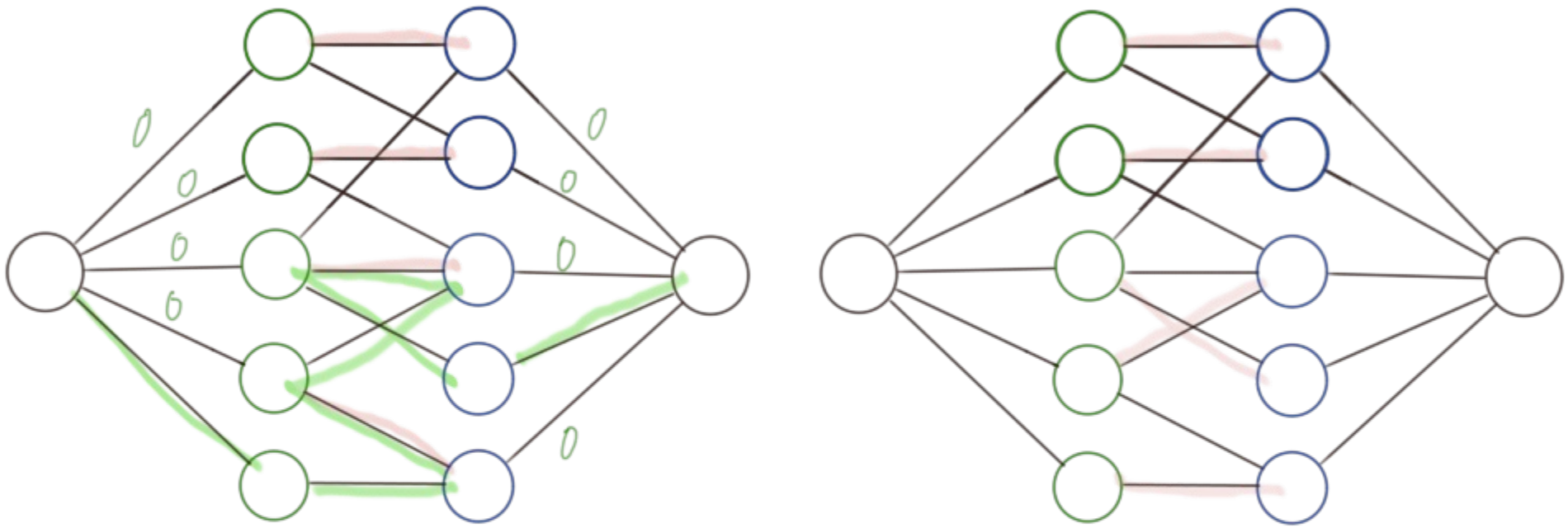

Any new path crosses the border

x times ->

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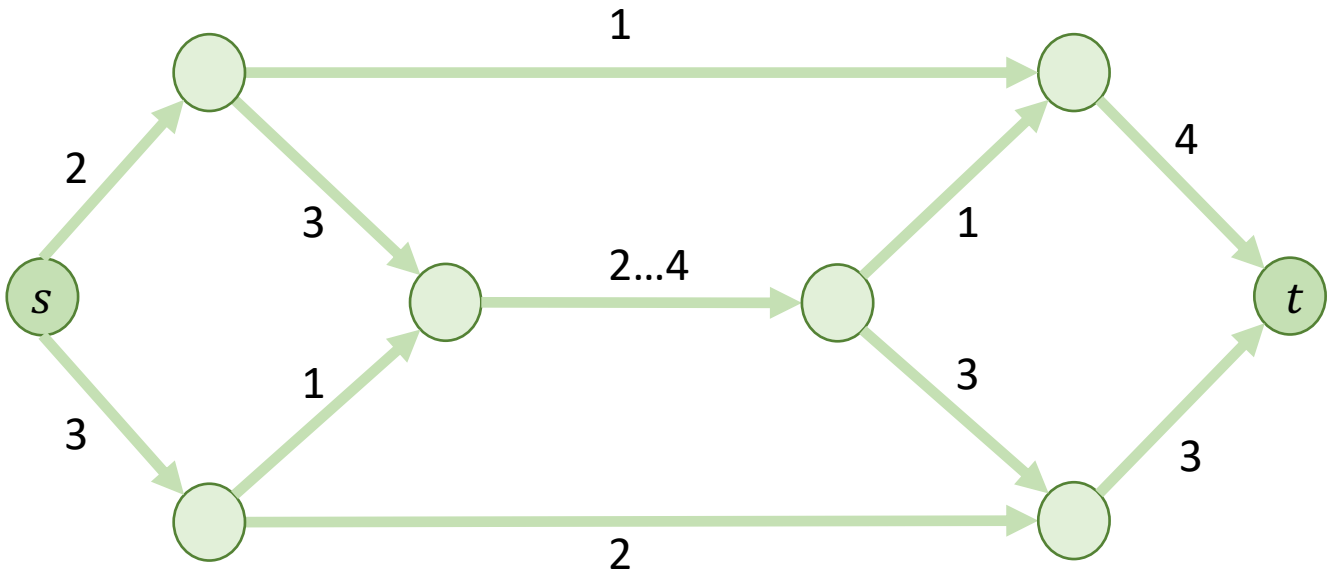
Minimum Capacity in Circulations

Suppose that some edges also have a minimum capacity.

Example: The middle edge must have a flow of at least 2 and at most 4.



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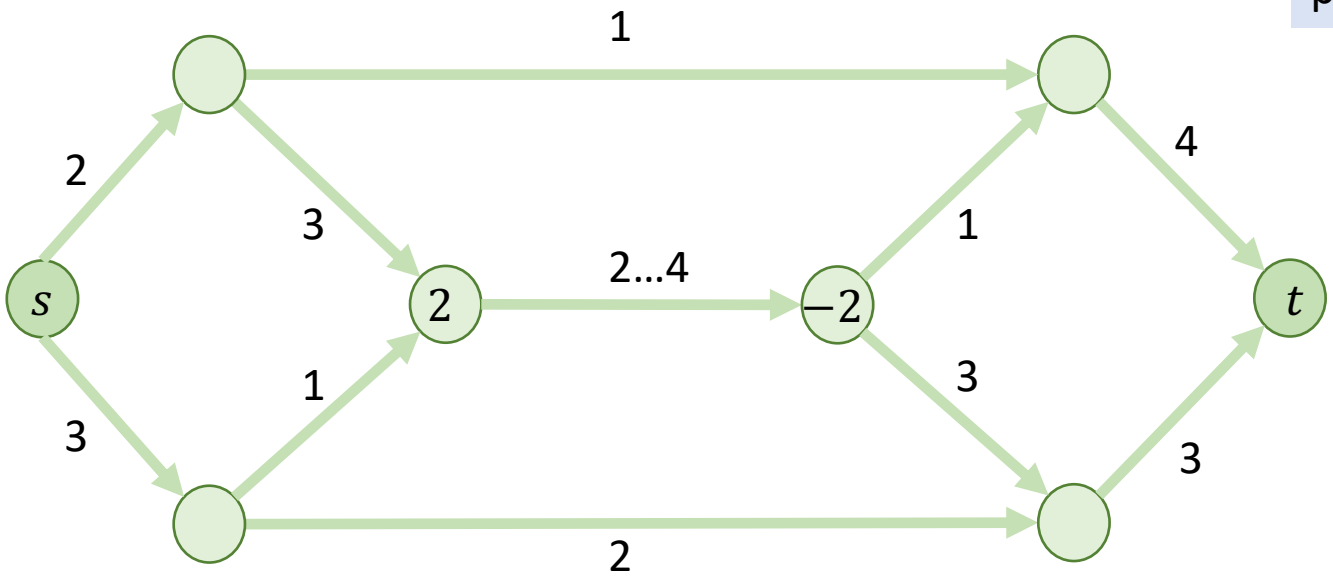

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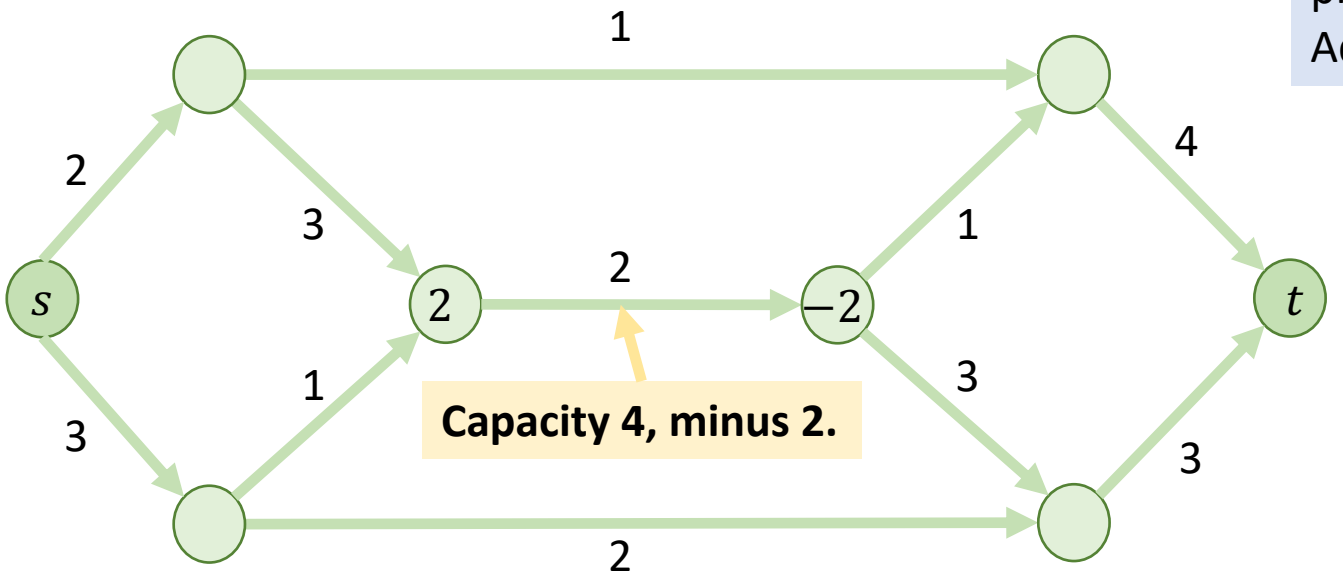

Impose a demand upstream, and produce the required amount downstream. Adjust the capacity to reserve space.

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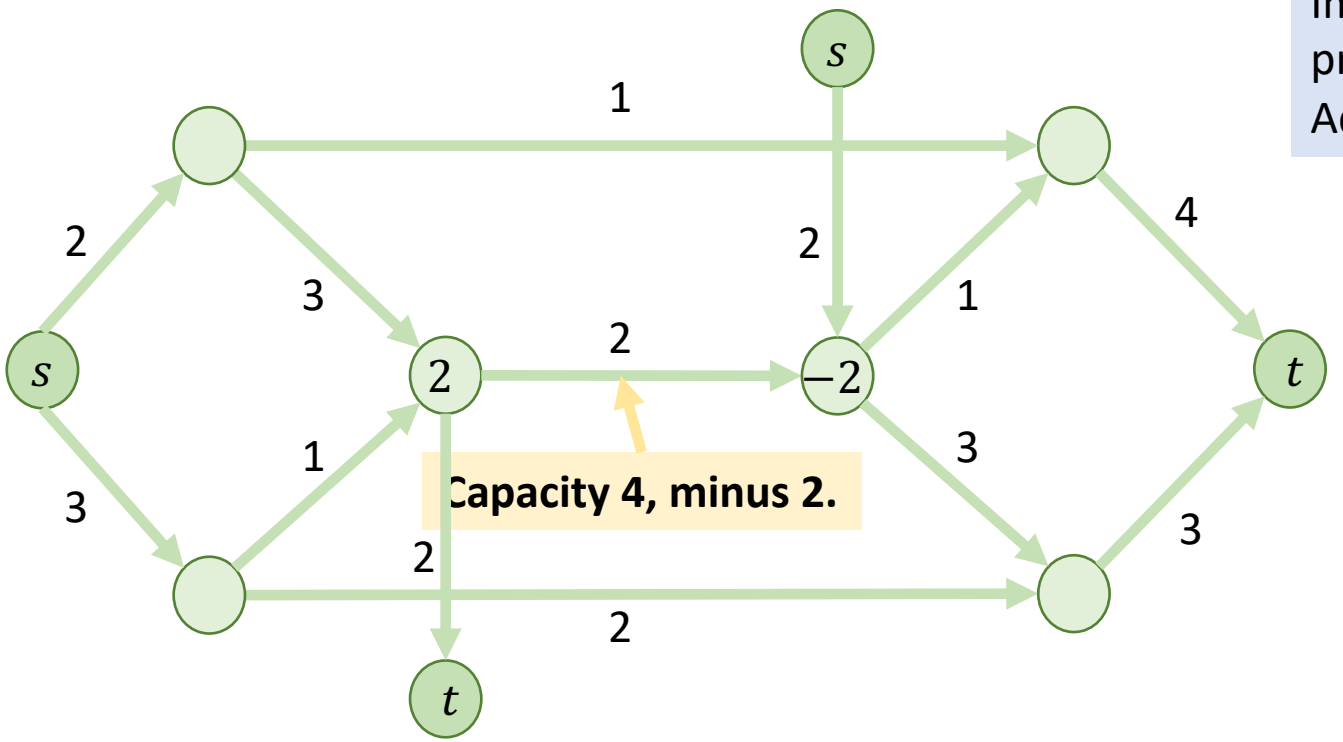

Impose a demand upstream, and produce the required amount downstream. Adjust the capacity to reserve space.

Minimum Capacity in Circulations

Suppose that some edges also have a minimum capacity.

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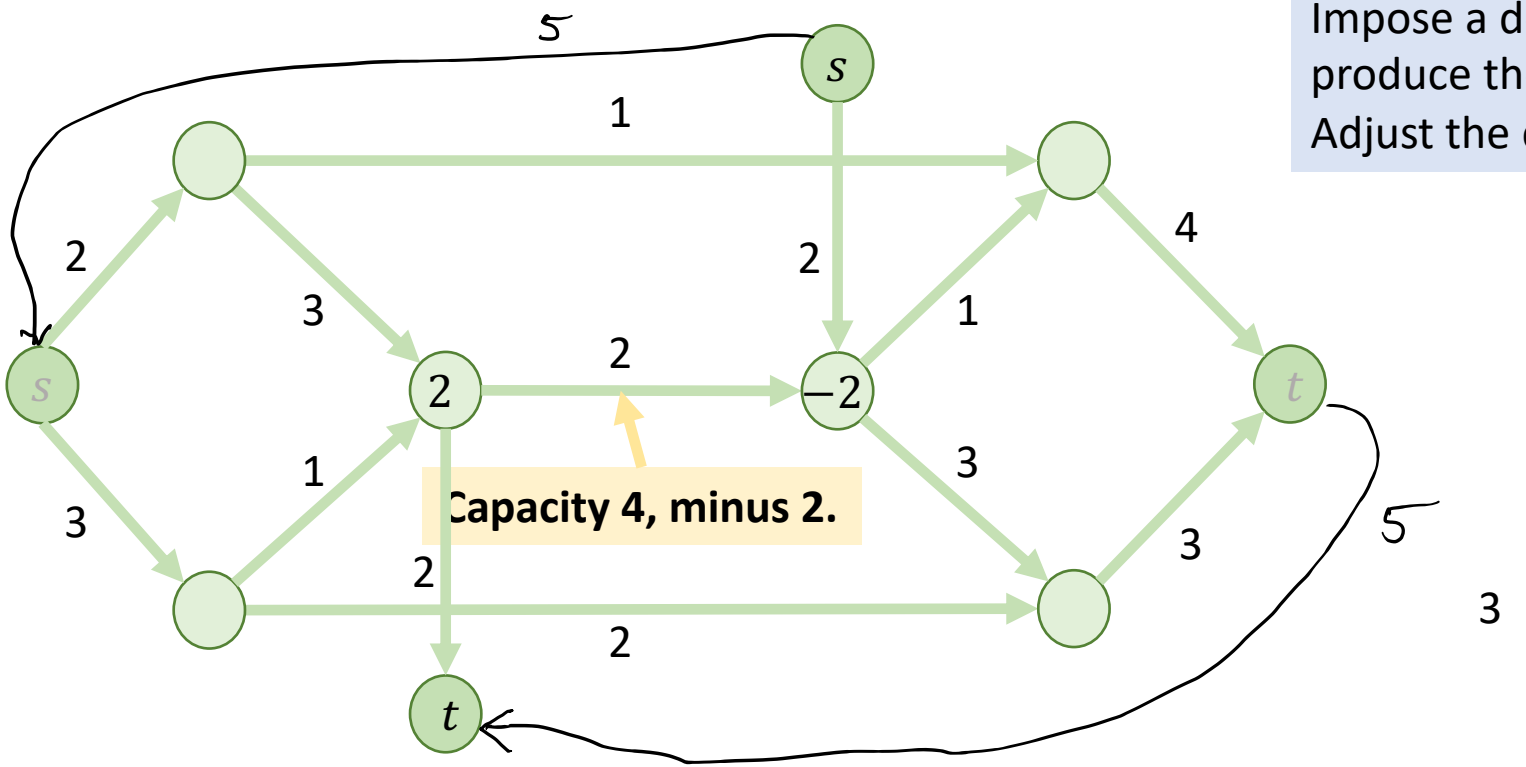

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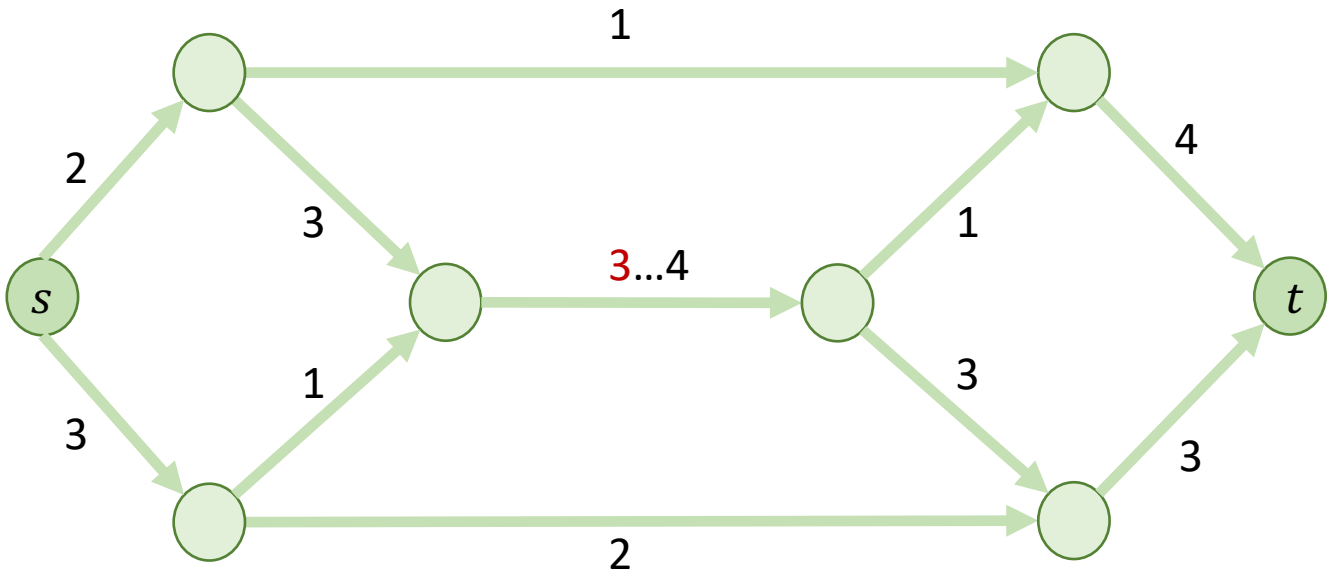

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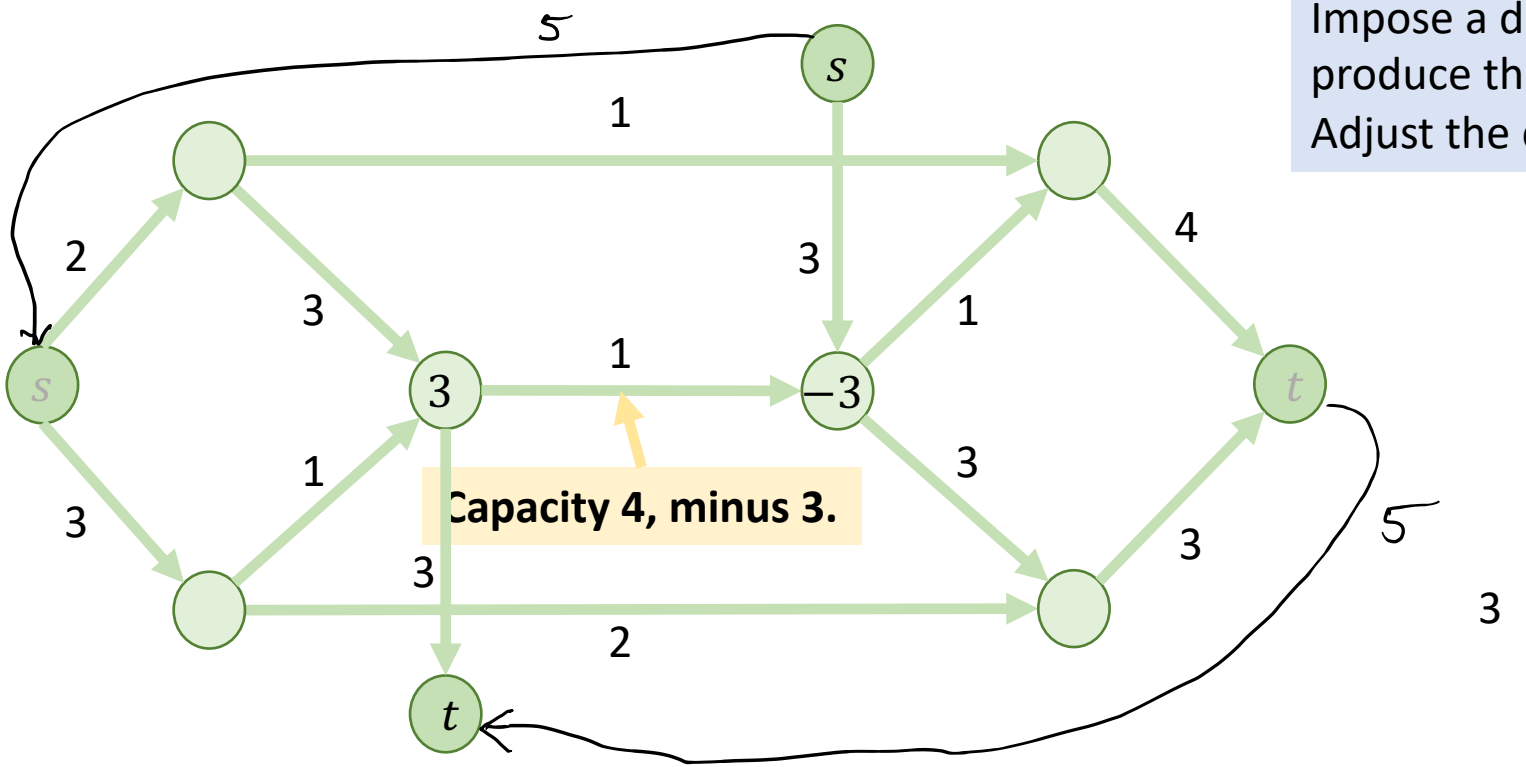

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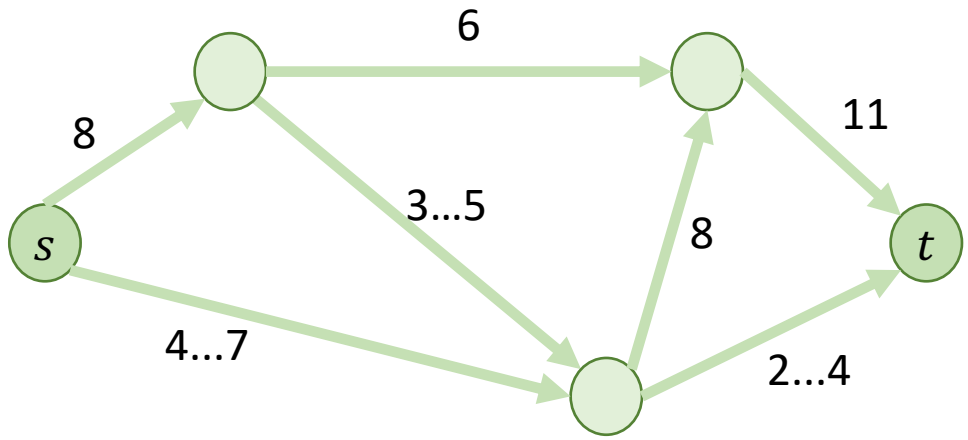
Min Capacity, Multiple edges

It’s likely that we see graphs with **multiple min** capacities



We use a similar approach.

- Fill in demands

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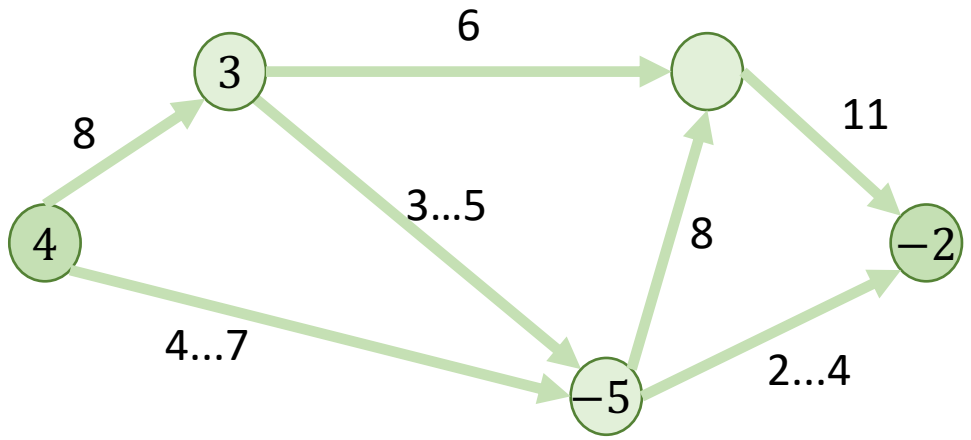

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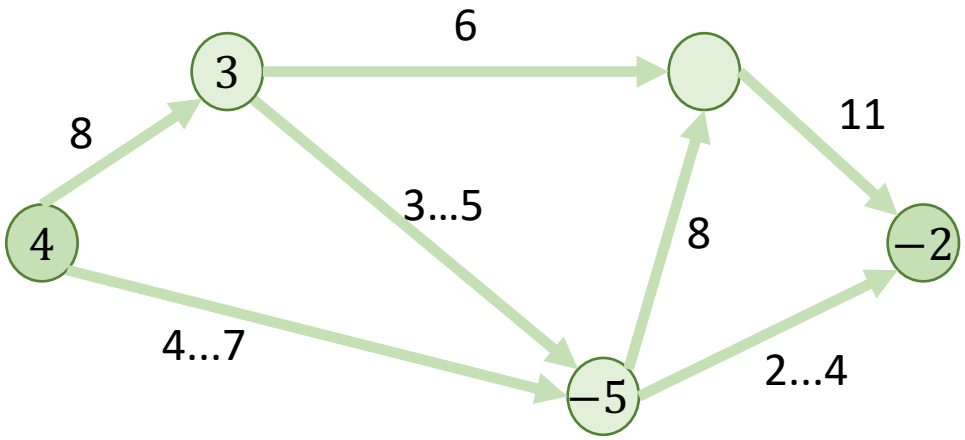

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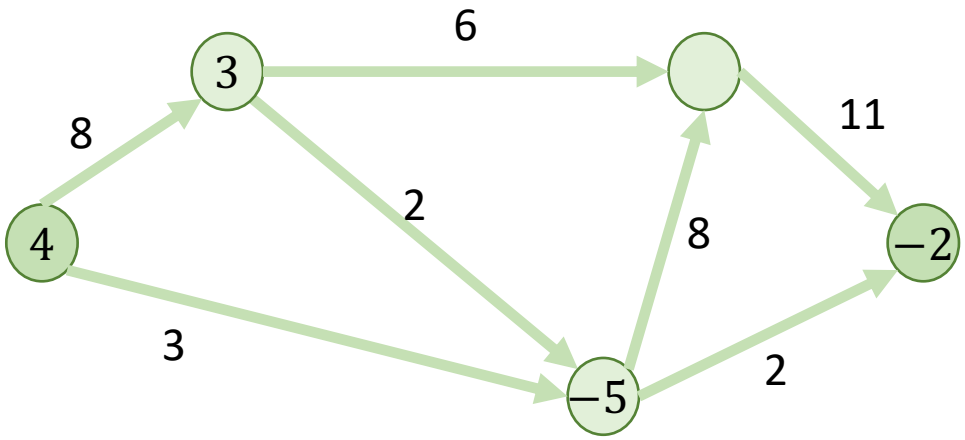

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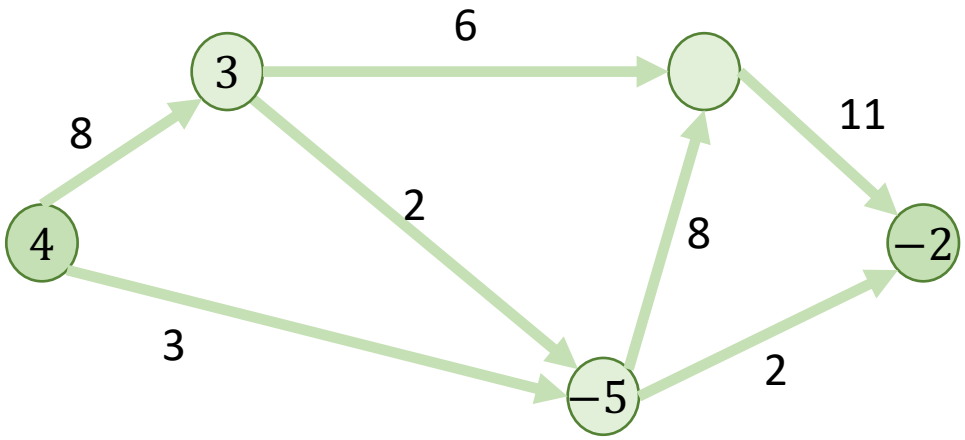

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We use a similar approach.

- Fill in demands
- Adjust capacities
- Create new sources/sinks

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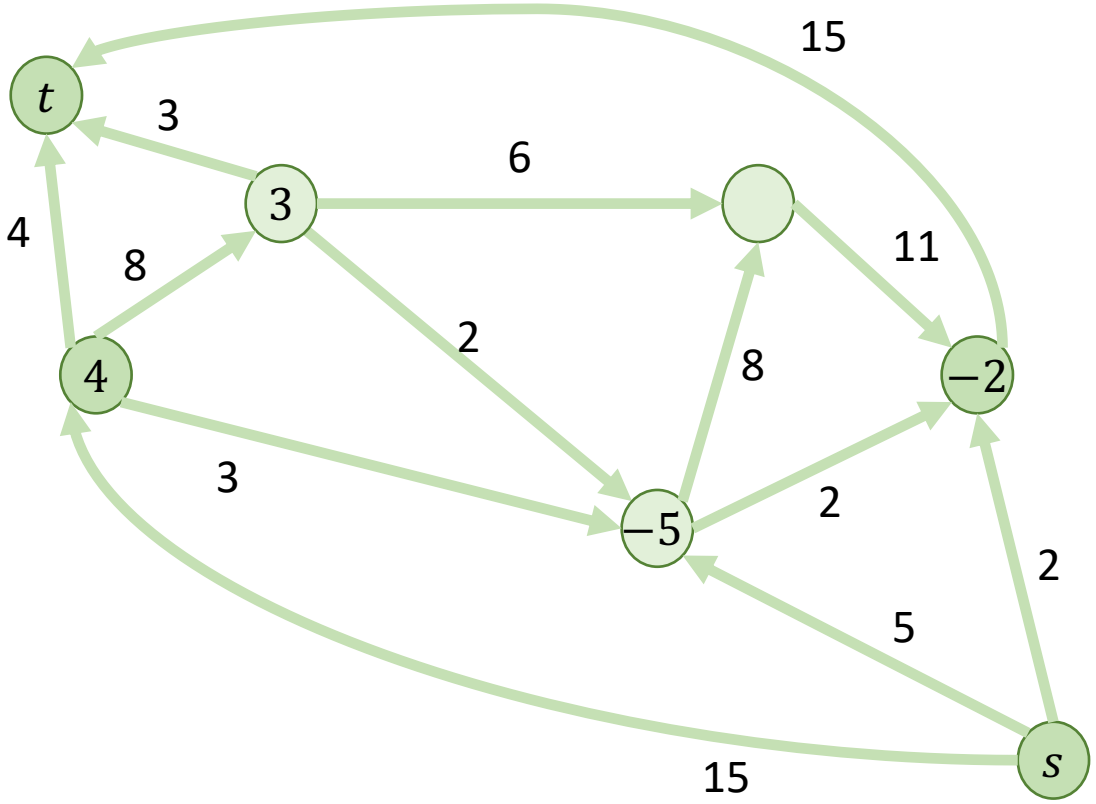

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Flow Networks for Airline Schedules

We will now use **max flow** algorithms to plan **airline schedules**.

Given:

- A fleet of k **airplanes**.
- A set of **flight routes** (source + destination + times).

Goal:

- Find a **flight plan** which determines which planes should fly which routes.
- Every route **must be serviced** by some plane.

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This is an interesting problem, because although we frame it as max flow, we are **not interested in maximizing flow** as much as we are interested in **satisfying minimums**.

Airline Schedule, Network Setup

- Our fleet has k airplanes.
- Every flight must be serviced.
- Airplanes can begin the day anywhere.
- Airplanes can end the day anywhere.
- A plane can fly its next leg out of the same airport at a later time.

Airline Schedule, Network Setup

- Our fleet has k airplanes.

  The source has a **capacity of k**.

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Airline Schedule, Network Setup

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  Edge from the source to the start of **every flight**.

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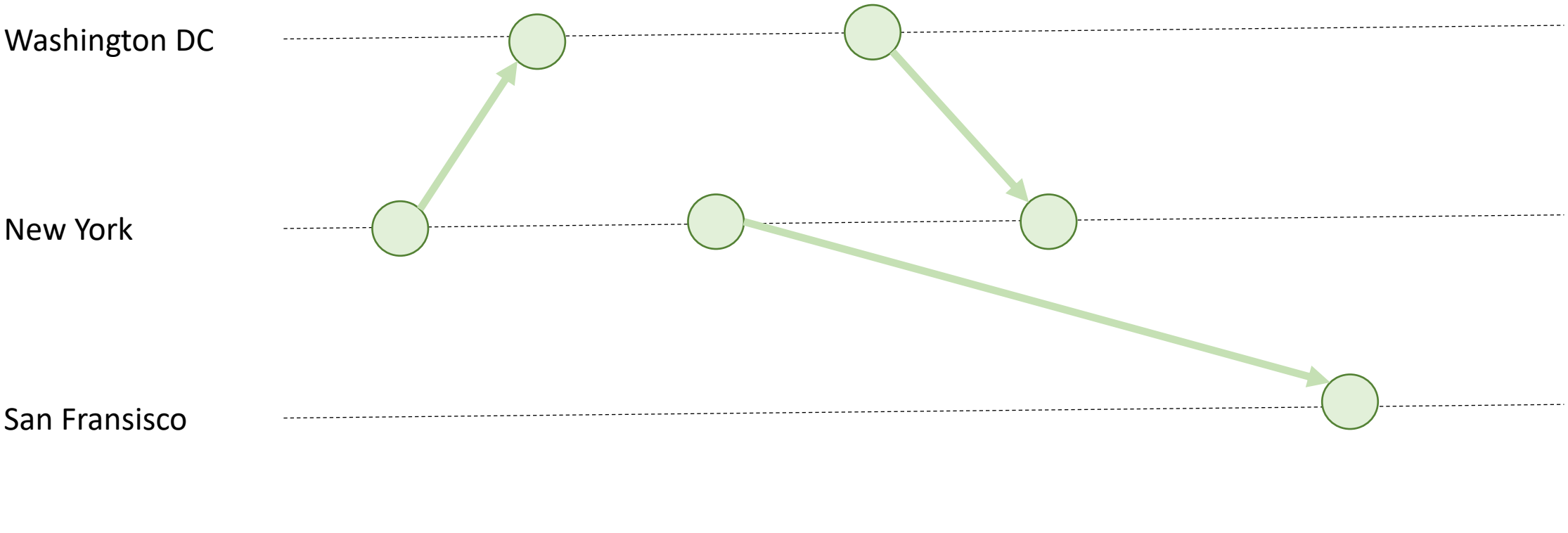
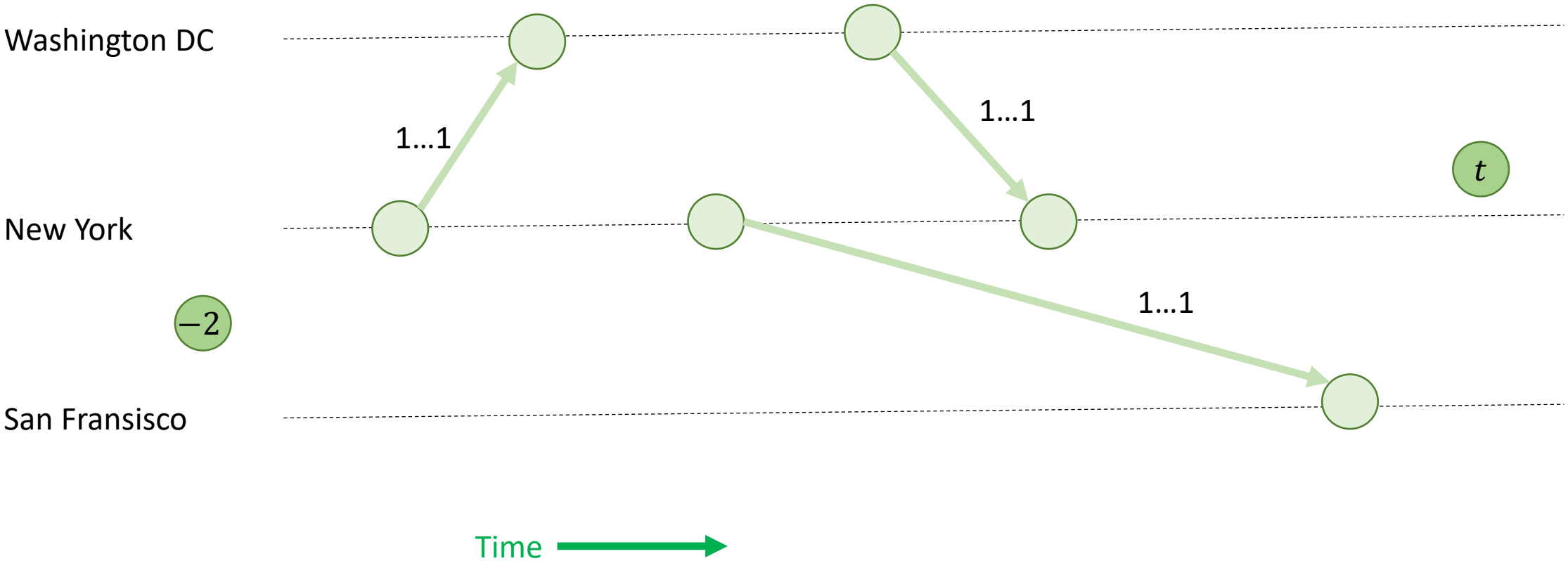
  Capacity 1 edges from the **end of any flight** to the **start of all other flights** at the same airport + later time.

Airline Schedule, Example



Assume $k = 2$ airplanes.

Airline Schedule, Example



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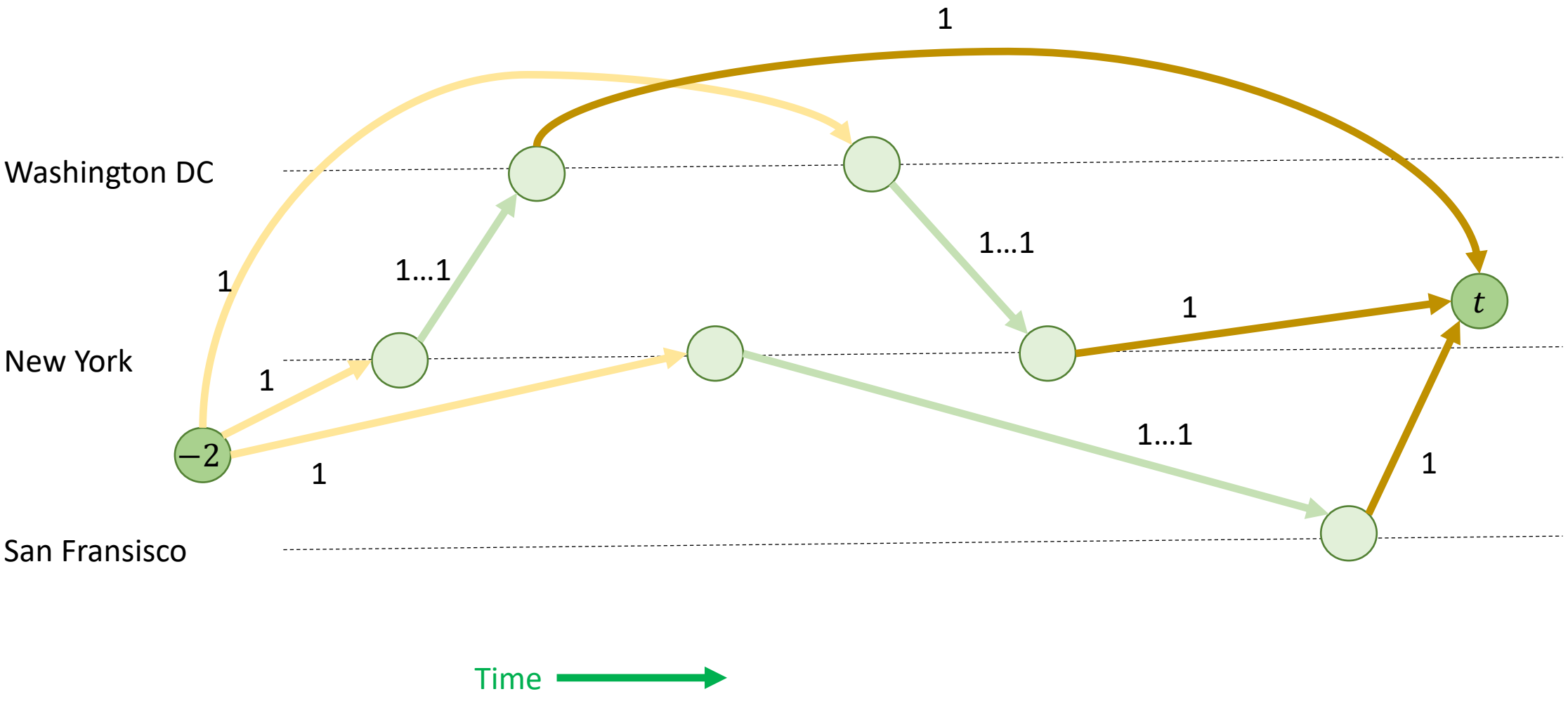

Airline Schedule, Example