Runtime of Ford-Fulkerson

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- Base case: they begin at 0.
- Inductive Hypothesis: Let $f_i(e)$ be the flow on edge e after i iterations. Assume $\forall e \in E, f_i(e)$ is an integer.
- In iteration $i + 1$ we modify each flow along the chosen path by some bottleneck value, which is either $c(e) - f_i(e)$ if the bottleneck is a forwards edge, or $f_i(e)$ if it is a backwards edge. Since all capacities are integer values, by the inductive hypothesis, both of these are integer values.

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- The new path augmentation P, starts at s.
- It never returns to s, because s has no incoming edges.
- The value of the flow leaving s increases by the bottleneck along P.

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Let $C = \sum_{e \text{ out of } s} c_e$. The algorithm terminates after at most C iterations.

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Let $C = \sum_{e \text{ out of } s} c_e$. The algorithm terminates after at most C iterations.

This follows immediately from the previous two claims. Note that if the algorithm hasn't already halted, then after the Cth iteration, the edges out of s must be saturated, and there cannot be any paths from s to t in the residual graph anymore.

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Runtime in each of the (at most) C iterations:
$O(m + n)$ for path finding.
$O(n)$ for augmenting the path, and the residual graph.

If we assume the graph is connected, $m \geq n - 1$, so this is $O(Cm)$.

Correctness of Ford-Fulkerson

Claim: Let f be any s-t flow, and let (A, B) be any s-t cut.
Then $v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$

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$$\begin{aligned} v(f) &= f^{\text{out}}(s) \\ &= \sum_{v \in A} f^{\text{out}}(v) - \sum_{v \in A} f^{\text{in}}(v) \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \end{aligned}$$

- The first equality is our definition of the value of a flow.
- The second equality holds by conservation of flow: for every $v \in A$ other than s (note, $t \notin A$), $f^{\text{out}}(v) = f^{\text{in}}(v)$, so we're simply adding a bunch of 0s to $f^{\text{out}}(s)$.
- The third equality is a bit subtle. Consider 4 types of edges, $\hat{e} = (x, y)$. If $x, y \in A$, then $f(\hat{e})$ appears in both $\sum_{v \in A} f^{\text{out}}(v)$ and $\sum_{v \in A} f^{\text{in}}(v)$, so they cancel each other out. If $x, y \notin A$, then $\hat{e}$ doesn't appear in either summation. If $x \in A, y \notin A$, then $f(\hat{e})$ appears only in the first sum, and finally if $x \notin A, y \in A$, it only appears in the second sum.

Intuitively, this means we can measure the value of the flow by measuring how much flow goes across any cut.

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Claim: Let f be any s-t flow, and let (A, B) be any s-t cut.
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$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\
&\leq \sum_{e \text{ out of } A} f(e) \\
&\leq \sum_{e \text{ out of } A} c(e) \\
&= c(A, B)
\end{aligned}
$$

Intuitively, this means that the value of *every* flow is less than or equal to the capacity of *every* cut. In particular, the value of the *maximum* flow is at most the capacity of the *minimum* cut.

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Ford-Fulkerson finds the maximum flow.

Let $\overline{f}$ be the flow output by Ford-Fulkerson on graph $G = (E, V)$.
Let $G_{\overline{f}}$ be the residual graph that results from executing the algorithm.
Let $A^* \subset V$ be the set of nodes reachable from s in $G_{\overline{f}}$.
We claim that (A^*, B^*) is an s-t cut such that $v(\overline{f}) = c(A^*, B^*)$.

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Claim: (A^*, B^*) is an s-t cut.
Clearly it is a partition of the vertices, since "reachability" is a binary property.
$s \in A^*$, since s is always reachable form itself.
$t \in B^*$, because the algorithm terminates when there is no s-t path in $G_{\overline{f}}$.

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Claim: for $e = (u, v) \in E$, with $u \in A^*, v \in B^*$, $f(e) = c(e)$.
Suppose to the contrary. Then there is a forward edge in $G_{\overline{f}}$ from u to v, contradicting the assumption that $v \in B^*$. (Since we have $s \rightsquigarrow u \in G_{\overline{f}}$, adding e gives a path to v.)

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Claim: for $e = (u, v) \in E$, with $u \in A^*, v \in B^*$, $f(e) = c(e)$.
Claim: for $e = (u, v) \in E$, with $u \in B^*, v \in A^*$, $f(e) = 0$.
Suppose to the contrary. Then there is a backward edge in $G_{\overline{f}}$ from v to u, contradicting the assumption that $u \in B^*$. (Since we have $s \rightsquigarrow v \in G_{\overline{f}}$, adding e gives a path to u.)

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Claim: for $e = (u, v) \in E$, with $u \in B^*, v \in A^*$, $f(e) = 0$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A^*} f(e) - \sum_{e \text{ into } A^*} f(e) \\ &= \sum_{e \text{ out of } A^*} c(e) - 0 \\ &= c(A^*, B^*) \end{aligned}$$